A 3D Anisotropic Diffusion scheme on ALE-AMR grids

J. BREIL, P. JACQ

CELIA, Université Bordeaux
breil@celia.u-bordeaux1.fr

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Context and objectives

Context

- We are interested in the modelisation of Laser/Plasma interaction
- We need 2 numerical bricks : A Lagrangian Hydrodynamics scheme and a Diffusion scheme
- One of the effect we want to take into account is the magnetic fields (imposed or self-generated)
- It is then mandatory to work in 3D geometries
- In terms of diffusion scheme we need to be able to deal with anisotropic non-symetric diffusion tensors (Braginskii model)
- In 3D the computational cost grows quickly : We want to obtain good accuracy without using too much cells

Objectives of this talk

Create accurate Finite Volume methods on unstructured grids for solving anisotropic diffusion.

- Develop the CCLADNS scheme in 3D.
- Gain accuracy through Adaptative Mesh Refinement (AMR) techniques.

2D radiation hydrodynamic code CHIC

Code CHIC (Code d'Hydrodynamique et d'Implosion du Celia)

Two Temperature Model (Te, Ti)

Cell Centered Lagrangian Scheme
Arbitrary Lagrangian Eulerian (ALE)
2D planar and axisymmetric

EOS

Electrons: TF, Ions: Cowan
QEOS, More, SESAME Tables

Thermonuclear combustion

in diffusion approximation

Electronic Transport

- Spitzër-Härm with Flux limitation
- Braginskii (B)
- Non local (NL) or B+NL

Cell centered diffusion scheme for isotropic or anisotropic media

Resistive MHD

Transverse MHD
Nernst transport

Opacities

Screened hydrogenic AA

3D laser ray tracing

refraction and collisional absorption

M1 transport Model

electrons, protons and ions

2D radiative transport

steady state multi-group
LTE or NLT opacities

J. Breil, S. Galera, P.-H. Maire, *Multi-Material ALE computation in Inertial Confinement Fusion code CHIC*, Computers and Fluids, Volume 46, Issue 1, Pages 161-167 (2011)

Braginskii Electron heat transport model

Two-temperature model

$$\rho \frac{d\varepsilon_{\mathrm{e}}}{dt} - \nabla \cdot (\boldsymbol{q_e}) = \Omega_{\mathrm{ei}}(T_{\mathrm{i}} - T_{\mathrm{e}}),$$
$$\rho \frac{d\varepsilon_{\mathrm{i}}}{dt} - \nabla \cdot (\lambda_{\mathrm{i}} \boldsymbol{\nabla} T_{\mathrm{i}}) = \Omega_{\mathrm{ei}}(T_{\mathrm{e}} - T_{\mathrm{i}}).$$

Electron heat flux $\boldsymbol{q_e}$

For the Braginskii model the electron heat flux in the plane orthogonal to the magnetic field $\boldsymbol{B}$ reads :

$$\boldsymbol{q_e} = -\lambda_{\perp} \boldsymbol{\nabla} T_e - \lambda_{\wedge} \boldsymbol{b} \times \boldsymbol{\nabla} T_e,$$

where $\boldsymbol{b} = \frac{\boldsymbol{B}}{|\boldsymbol{B}|}$, T_e is the electron temperature, $\lambda_{\perp}$ and $\lambda_{\wedge}$ are the heat conduction coefficients. We write the electronic heat flux in the form $\boldsymbol{q_e} = -\mathbb{K}\boldsymbol{\nabla} T_e$, where $\mathbb{K}$ is a tensorial heat conduction coefficient:

$$\mathbb{K} = \begin{pmatrix} \lambda_{\perp} & -b_z \lambda_{\wedge} \\ b_z \lambda_{\wedge} & \lambda_{\perp} \end{pmatrix}.$$

- $\mathbb{K}$ is not symmetric and this non-symmetry of the conductivity tensor is a consequence of the presence of the magnetic field.
- This behavior is known as the Righi-Leduc effect. It rotates the heat flux vector without changing its absolute value.

S. I. Braginskii, *in Review of Plasma Physics*, edited by M. A. Leontovitsh 1965;**1**: 205–311.

Review : 3D diffusion scheme

Needs

- Handle complex geometries : unstructured meshes
- Handle multi-materials : cell-centered (material interfaces defined by the mesh)

Some classical Finite Volume schemes

- TPFA - Two Point Flux Approximation
- MPFA - Multi Point Flux Approximation (Aavatsmark JSC 1998)
- MFD - Mimetic Finite Differences (Shashkov JCP 1997)
- DDFV - Discrete Duality Finite Volume (Hermeline JCP 2000)

CCLAD and CCLADNS schemes

- CCLAD scheme familly : Cell-Centered LAgrangian Diffusion
- CCLAD 2D : Breil-Maire JCP 2007, CCLADNS 2D and 2D axi : Maire-Breil JCP 2012
- CCLAD 3D and parallel implementation (MPI) : Jacq-Maire CiCP 2014

New developpements presented in this talk

- Extension of the CCLADNS scheme to 3D geometries
- Handling of Adaptative Mesh Refinement (AMR) grids

Developpement of a 3D diffusion scheme

Diffusion equation in the domain $\mathcal{D} \subset \Re^3$

$$
\begin{aligned}
&\rho C \frac{\partial T}{\partial t} + \nabla \cdot \boldsymbol{q} = 0 && (\boldsymbol{x}, t) \in \mathcal{D} \times [0, \mathfrak{T}], \\
&T(\boldsymbol{x}, t) = T^0(\boldsymbol{x}), && \boldsymbol{x} \in \mathcal{D}, \\
&T(\boldsymbol{x}, t) = T^\star(\boldsymbol{x}, t), && \boldsymbol{x} \in \partial\mathcal{D}_D, \text{ Dirichlet}, \\
&\boldsymbol{q}(\boldsymbol{x}, t) \cdot \boldsymbol{n} = q_N^\star(\boldsymbol{x}, t), && \boldsymbol{x} \in \partial\mathcal{D}_N, \text{ Neumann}, \\
&\alpha T(\boldsymbol{x}, t) + \beta \boldsymbol{q}(\boldsymbol{x}, t) \cdot \boldsymbol{n} = q_R^\star(\boldsymbol{x}, t), && \boldsymbol{x} \in \partial\mathcal{D}_R, \text{ Robin}
\end{aligned}
$$

Fourier's law for the heat flux

$$\boldsymbol{q} = -\mathbb{K} \boldsymbol{\nabla} T,$$

where $\mathbb{K}$ is the conductivity tensor

Continuity of the normal flux at the interfaces

$$\boldsymbol{q}_1 \cdot \boldsymbol{n}_{12} = \boldsymbol{q}_2 \cdot \boldsymbol{n}_{12}.$$

$\boldsymbol{n}_{12}$ is the unit normal at the interface between the materials 1 and 2.

3D diffusion scheme : Sub-cell discretization

Finite volume method

We integrate $\rho C_v \dfrac{\partial T}{\partial t} + \nabla \cdot \boldsymbol{q} = 0$ over a cell ω_c

We obtain $m_c C_{vc} \dfrac{d}{dt} T_c + \displaystyle\int_{\partial\omega_c} \boldsymbol{q} \cdot \boldsymbol{n} = 0$

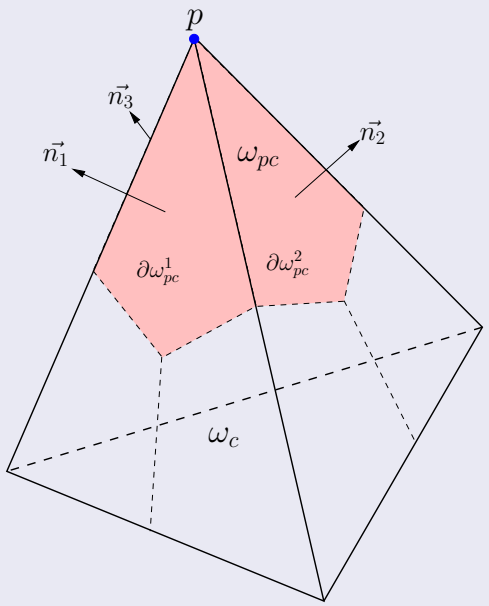
Flux discretization

We will introduce a sub-cell discretization

$$\omega_c = \bigcup_{p \in \mathcal{P}(c)} \omega_{pc}$$

where $\mathcal{P}(c)$ is the set of nodes in cell c

Notations around a point p



We introduce the Sub-Face Normal Flux

$$q^f_{pc} = \frac{1}{S^f_{pc}} \int_{\partial\omega^f_{pc}} \boldsymbol{q} \cdot \boldsymbol{n} \, ds$$

The flux writes

$$\int_{\partial\omega_c} \boldsymbol{q} \cdot \boldsymbol{n} = \sum_{f \in \mathcal{F}(p,c)} S^f_{pc} q^f_{pc}$$

where

- $f \in \mathcal{F}(p, c)$ the set of sub-faces in cell c impinging at node p
- $S^f_{pc} = |\partial\omega^f_{pc}|$
- The sub-faces are triangulated to obtain $\boldsymbol{n}$ and S^f_{pc}

CCLADNS 3D : Key ingredients

Introduction of auxiliary unknowns : sub-faces Temperatures

$$\overline{T}_p^f = \frac{1}{S_p^f} \int_{\partial\omega_{pc}^f} T(\boldsymbol{x}, t)\, \mathrm{d}s$$

We want to express q_{pc}^f under the form $q_{pc}^f = h_{pc}^f(\overline{T}_{pc}^1 - T_c, , \cdots, \overline{T}_{pc}^f - T_c, \cdots, \overline{T}_{pc}^F - T_c)$

Piecewise linear approximation of the temperature over the cell ω_c

$$T_h(\boldsymbol{x}) = T_c - \mathbb{K}_c^{-1} \boldsymbol{q}_c \cdot (\boldsymbol{x} - \boldsymbol{x}_c).$$

Using $\boldsymbol{x}_{pc}^f = \overline{\boldsymbol{x}}_p^f - \boldsymbol{x}_c$ where $\overline{\boldsymbol{x}}_p^f$ denote the vector position of the sub-face centroid we have

$$\overline{T}_p^f - T_c = -\mathbb{K}_c^{-1} \boldsymbol{q}_{pc} \cdot \boldsymbol{x}_{pc}^f.$$

The decomposition of $\boldsymbol{q}_{pc}$ in terms of its normal components writes

$$\mathbb{J}_{pc}^t \boldsymbol{q}_{pc} = \begin{pmatrix} q_{pc}^1 \\ q_{pc}^2 \\ q_{pc}^3 \end{pmatrix}$$

Where $\mathbb{J}_{pc} = [\boldsymbol{n}_{pc}^1, \boldsymbol{n}_{pc}^2, \boldsymbol{n}_{pc}^3]$. Then we have

$$\overline{T}_p^f - T_c = -\mathbb{K}_c^{-1} \mathbb{J}_{pc}^{-t} \begin{pmatrix} q_{pc}^1 \\ q_{pc}^2 \\ q_{pc}^3 \end{pmatrix} \cdot \boldsymbol{x}_{pc}^f.$$

Introduction of auxiliary unknowns : sub-faces Temperatures

Using the identity

$$\boldsymbol{x}^f_{pc} = \mathbb{J}^{-t}_{pc}\mathbb{J}^t_{pc}\boldsymbol{x}^f_{pc} = \mathbb{J}^{-t}_{pc}\begin{pmatrix}\boldsymbol{x}^f_{pc}\cdot\boldsymbol{n}^1_{pc}\\ \boldsymbol{x}^f_{pc}\cdot\boldsymbol{n}^2_{pc}\\ \boldsymbol{x}^f_{pc}\cdot\boldsymbol{n}^3_{pc}\end{pmatrix}$$

We introduce the matrix $\mathbb{X}^{i,j}_{pc} = \boldsymbol{x}^i_{pc}\cdot\boldsymbol{n}^j$.

$$\overline{T}^f_p - T_c = -\mathbb{J}^{-1}_{pc}\mathbb{K}^{-1}_c\mathbb{J}^{-t}_{pc}\begin{pmatrix}q^1_{pc}\\ q^2_{pc}\\ q^3_{pc}\end{pmatrix}\cdot\begin{pmatrix}\mathbb{x}^{f,1}_{pc}\\ \mathbb{x}^{f,2}_{pc}\\ \mathbb{x}^{f,3}_{pc}\end{pmatrix}.$$

Using all the temperatures of the sub-faces and introducing the sub-cell conductivity tensor

$$\mathbb{K}_{pc} = \mathbb{J}^t_{pc}\mathbb{K}_c\mathbb{J}_{pc}$$

$$\begin{pmatrix}\overline{T}^1_p - T_c\\ \overline{T}^2_p - T_c\\ \overline{T}^3_p - T_c\end{pmatrix} = -\mathbb{X}_{pc}\mathbb{K}^{-1}_{pc}\begin{pmatrix}q^1_{pc}\\ q^2_{pc}\\ q^3_{pc}\end{pmatrix}.$$

Introduction of auxiliary unknowns : sub-faces Temperatures

We finally have

$$\begin{pmatrix} q^1_{pc} \\ q^2_{pc} \\ q^3_{pc} \end{pmatrix} = -\mathbb{K}_{pc}\mathbb{X}_{pc}^{-1} \begin{pmatrix} \overline{T}^1_p - T_c \\ \overline{T}^2_p - T_c \\ \overline{T}^3_p - T_c \end{pmatrix}.$$

Elimination of the auxiliary unknowns : Continuity conditions

- Normal flux is continuous on every sub-face : $q^f_{pc} + q^f_{pc'} = 0$
- Temperature is continuous on every sub-face : $\overline{T}^f_{pc} = \overline{T}^f_{pc'} = \overline{T}^f_p$

We can write a local system of equations around mesh nodes under the form $\mathbb{M}\overline{\boldsymbol{T}} = \mathbb{S}\boldsymbol{T}$.

3D diffusion scheme

Scheme final expression

$$m_c C_c \frac{d}{dt} T_c + \sum_{p \in \mathcal{P}(c)} \sum_{d \in \mathcal{C}(p)} \mathbb{\Gamma}^p_{c,d} (T_d - T_c) = B_c$$

With

$$\mathbb{\Gamma}^p = \widetilde{\mathbb{S}}^t \mathbb{M}^{-1} \mathbb{S}$$

The global system writes:

$$\mathcal{M} \mathcal{C}_v \frac{d\mathcal{T}}{dt} + \mathcal{A}\mathcal{T} = \mathcal{B}$$

Scheme properties

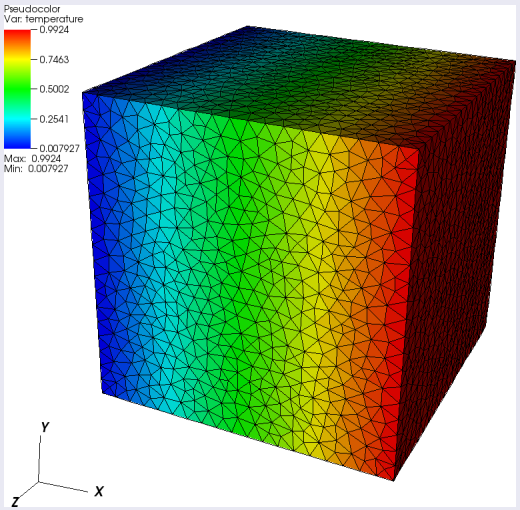
- Boundary conditions naturally taken into account in the local systems $\mathbb{M}\overline{\boldsymbol{T}} = \mathbb{S}\boldsymbol{T} + \boldsymbol{B}$
- The scheme uses a local stencil (sparse global matrices)
- Easily **parallelizable** (Jacq-Maire CiCP 2014)

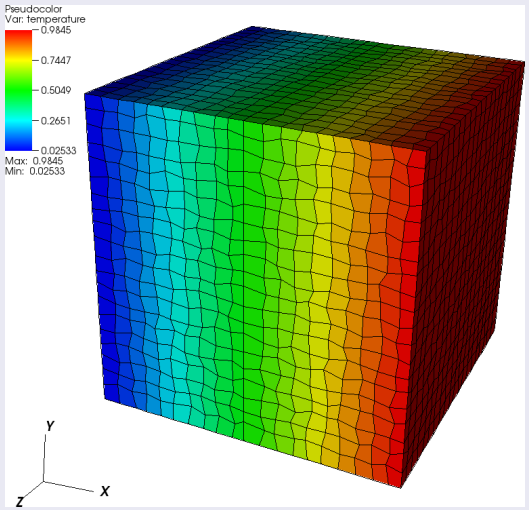
Time discretization

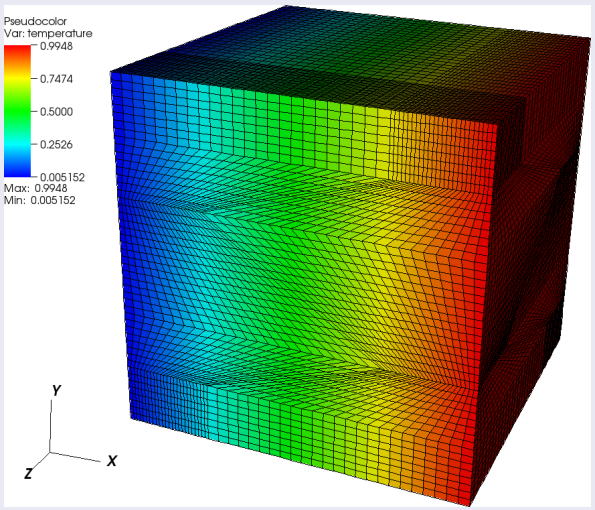
- Implicit scheme, first order in time
- Global system is solved using iterative methods (PETSc library)

Numerical results

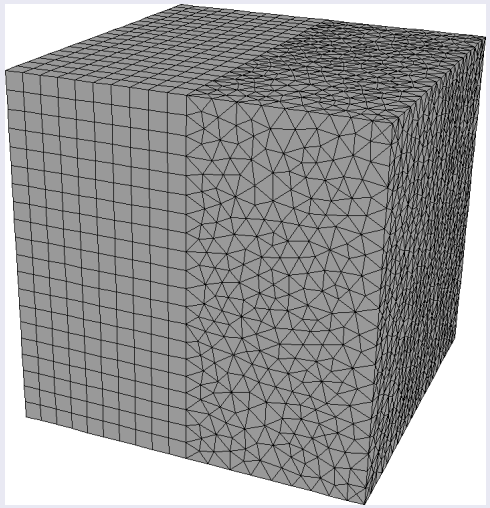
Preservation of linear fields with CCLADNS (Up to machine precision)

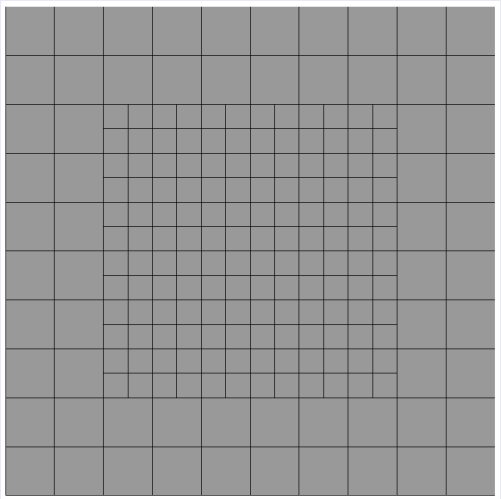


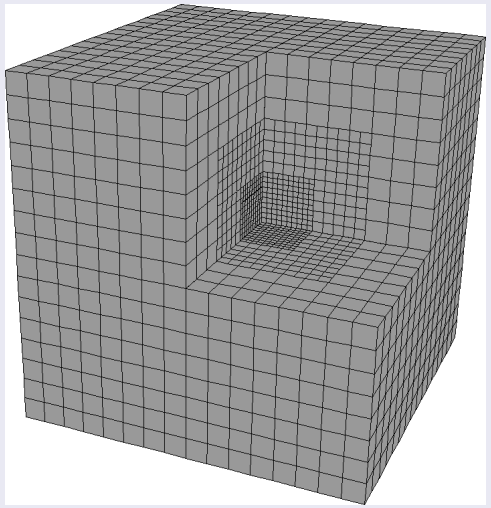




Some type of grids remains problematic





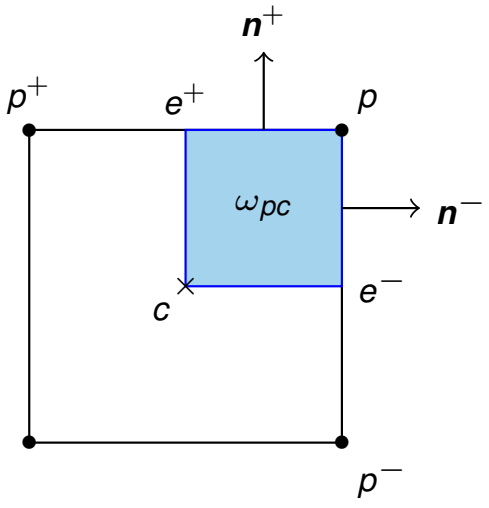


The CCLADNS scheme in 2D

In each subcell ω_{pc} the normal fluxes writes

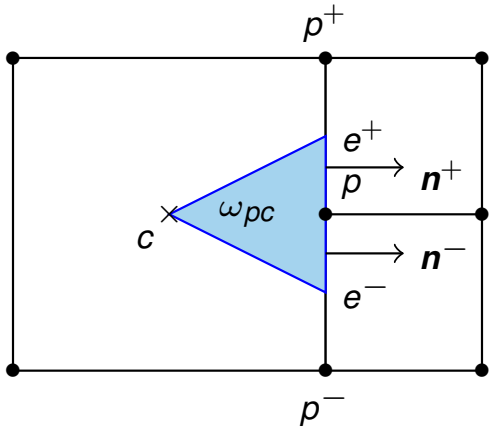
$$\begin{pmatrix} q_{pc}^- \\ q_{pc}^+ \end{pmatrix} = -\mathbb{K}_{pc}\mathbb{X}_{pc}^{-1}\begin{pmatrix} \bar{T}_p^- - T_c \\ \bar{T}_p^+ - T_c \end{pmatrix} = -\mathbb{L}_{pc}\begin{pmatrix} \bar{T}_p^- - T_c \\ \bar{T}_p^+ - T_c \end{pmatrix} \quad (1)$$

- ω_{pc} are always quadrilateral
- The matrices $\mathbb{K}_{pc}$ and $\mathbb{X}_{pc}$ depends on vectors $\boldsymbol{n}^-$ and $\boldsymbol{n}^+$
- The inverse of matrix $\mathbb{X}_{pc}$ exists iff $\boldsymbol{n}^-$ and $\boldsymbol{n}^+$ form a basis



The CCLADNS scheme in 2D on an AMR cell

- p is an AMR node, or hanging node, it lies on the edge $[p^-, p^+]$
- ω_{pc} degenerates to a triangle
- $\boldsymbol{n}^-$ and $\boldsymbol{n}^+$ are colinear, the inverse of $\mathbb{X}_{pc}$ does not exist
- Can we still express the normal fluxes under a form similar to (1) ?



Modification to the CCLADNS scheme to handle 2D AMR grids

Subcell splitting

- We can divide the subcell ω_{pc} in two subcells ω_{pc}^{-} and ω_{pc}^{+}
- In each subcells ω_{pc}^{-} and ω_{pc}^{+} the normal vectors are not colinear

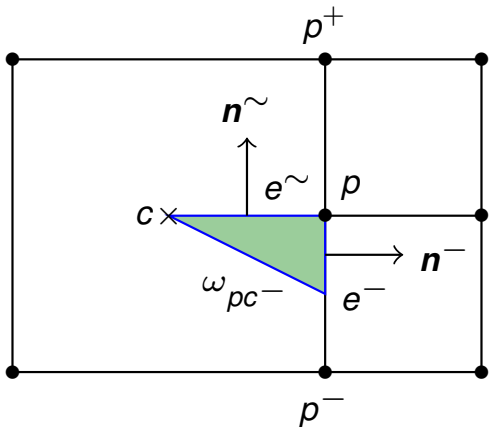
$$\begin{pmatrix} q_{pc}^{-} \\ q_{pc}^{\sim} \end{pmatrix} = -\mathbb{L}_{pc}^{-} \begin{pmatrix} \bar{T}_p^{-} - T_c \\ \bar{T}_p^{\sim} - T_c \end{pmatrix} \tag{2}$$

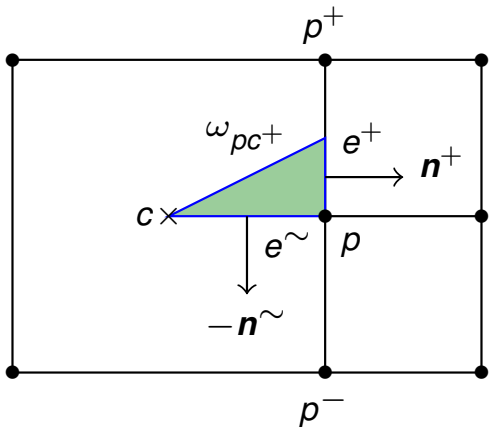
$$\begin{pmatrix} q_{pc}^{\sim} \\ q_{pc}^{+} \end{pmatrix} = -\mathbb{L}_{pc}^{+} \begin{pmatrix} \bar{T}_p^{\sim} - T_c \\ \bar{T}_p^{+} - T_c \end{pmatrix} \tag{3}$$

Using the continuity conditions on edge $e^{\sim}$ these reduce to

$$\begin{pmatrix} q_{pc}^{-} \\ q_{pc}^{+} \end{pmatrix} = -\mathbb{L}_{pc}^{\sim} \begin{pmatrix} \bar{T}_p^{-} - T_c \\ \bar{T}_p^{+} - T_c \end{pmatrix} \tag{4}$$

The matrix $\mathbb{L}_{pc}^{\sim}$ depends on vectors $\boldsymbol{n}^{-}$, $\boldsymbol{n}^{+}$ and $\boldsymbol{n}^{\sim}$.
The global system is then obtained using the usual methodology.





AMR grids in 3D geometries

The CCLADNS scheme in 3D

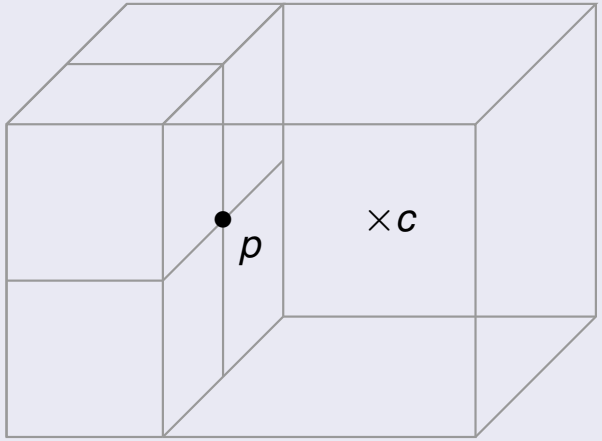
In each subcell ω_{pc} the normal fluxes writes

$$\begin{pmatrix} q^1_{pc} \\ q^2_{pc} \\ q^3_{pc} \end{pmatrix} = -\mathbb{K}_{pc}\mathbb{X}_{pc}^{-1} \begin{pmatrix} \bar{T}^1_p - T_c \\ \bar{T}^2_p - T_c \\ \bar{T}^3_p - T_c \end{pmatrix} = -\mathbb{L}_{pc} \begin{pmatrix} \bar{T}^1_p - T_c \\ \bar{T}^2_p - T_c \\ \bar{T}^3_p - T_c \end{pmatrix} \tag{5}$$

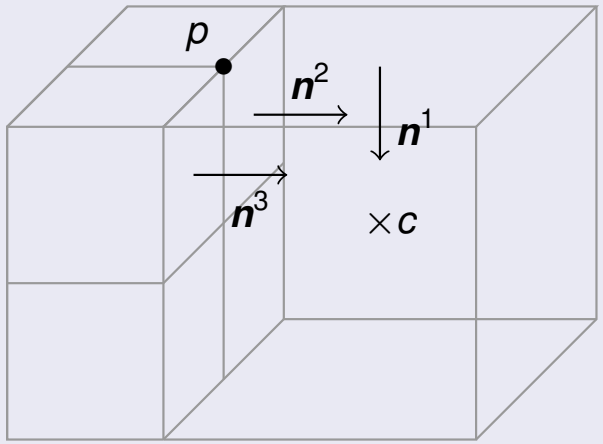
- Works only for 3 subfaces impinging at node p ($F_{pc} = 3$)
- The inverse of matrix $\mathbb{X}_{pc}$ exists iff $\boldsymbol{n}^1$,$\boldsymbol{n}^2$ and $\boldsymbol{n}^3$ form a basis

Troubles encountered on AMR grids

The "pyramidal" problem : $F_{pc} > 3$



Matrix $\mathbb{X}_{pc}$ can be singular:



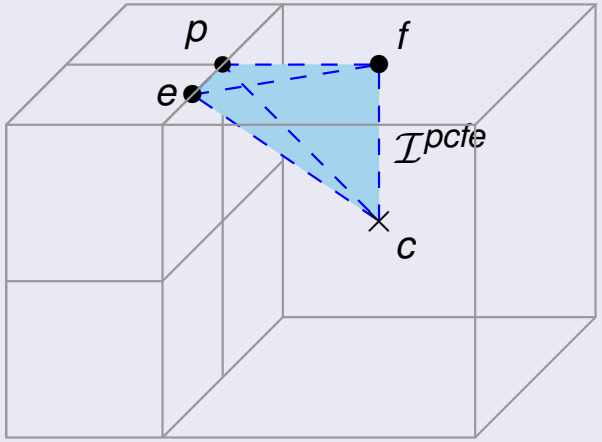
The vectors $\boldsymbol{n}^2$ and $\boldsymbol{n}^3$ are colinear.

Modification to the CCLADNS scheme to handle 3D AMR grids

Introducing the iota cells (Burton 1994)

A iota cell is the smallest simplex that allow us to divide a cell.
A iota $\mathcal{I}^{pcfe}$ is a tetrahedron described by 4 entities :

- a node p,
- a cell centroid c,
- a face barycenter f,
- the middle of an edge e.



It has the following properties:

- On every iota $\mathcal{I}^{pcfe}$ we have only 3 subfaces impinging at node p.
- On every iota $\mathcal{I}^{pcfe}$ the normal vectors $\boldsymbol{n}^1$,$\boldsymbol{n}^2$ and $\boldsymbol{n}^3$ form a basis.

This is perfect for writing CCLADNS!

Using iota cells as subcells in CCLADNS

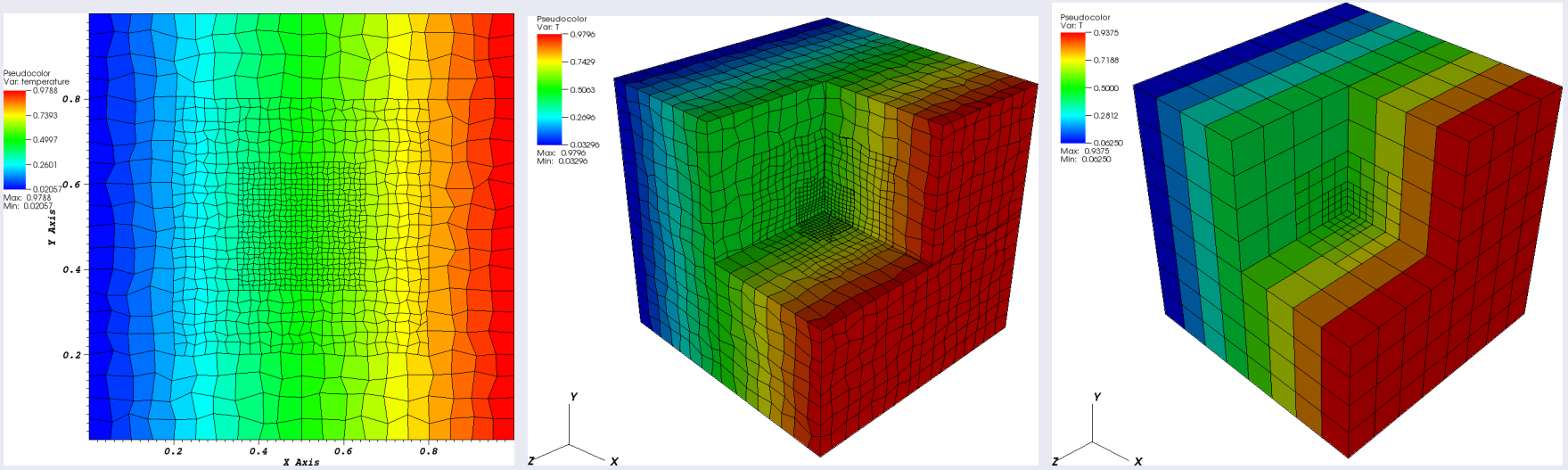
On every iota cell we are able to compute the normal fluxes (5).
The local nodal systems rewrites $\mathbb{M}^{\triangle}\,\bar{\boldsymbol{T}}^{\triangle} = \mathbb{S}^{\triangle}\,\boldsymbol{T}^{\triangle}$ with:

- $\bar{\boldsymbol{T}}^{\triangle}$ the vector of sub-face temperatures,
- $\boldsymbol{T}^{\triangle}$ the vector of sub-cell temperatures.

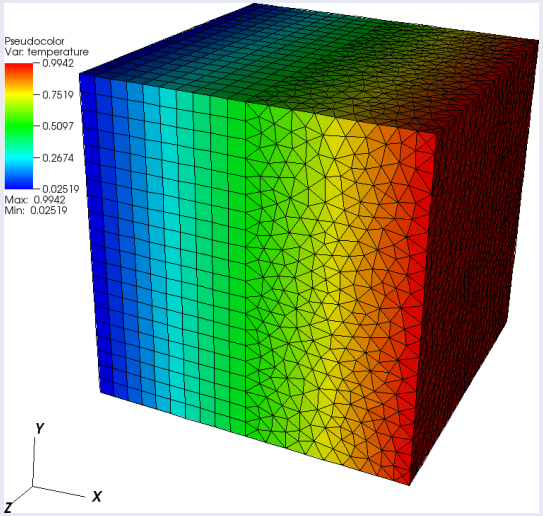
Finally the relation $\boldsymbol{T}^{\triangle} = \mathbb{P}\boldsymbol{T}$, with $\boldsymbol{T}$ the vector of cell temperatures, allows us to write the global system in terms of cell temperatures.

CCLADNS 2D/3D results

Linear fields preserved on AMR grids



Linear fields also preserved on hybrid grids : Hex-Tet and Pyramids



The CCLADS version demonstrates on these meshes a 1st order convergence rate only. With the maximal error always lying in the layer of pyramids.

2D AMR non-linear results

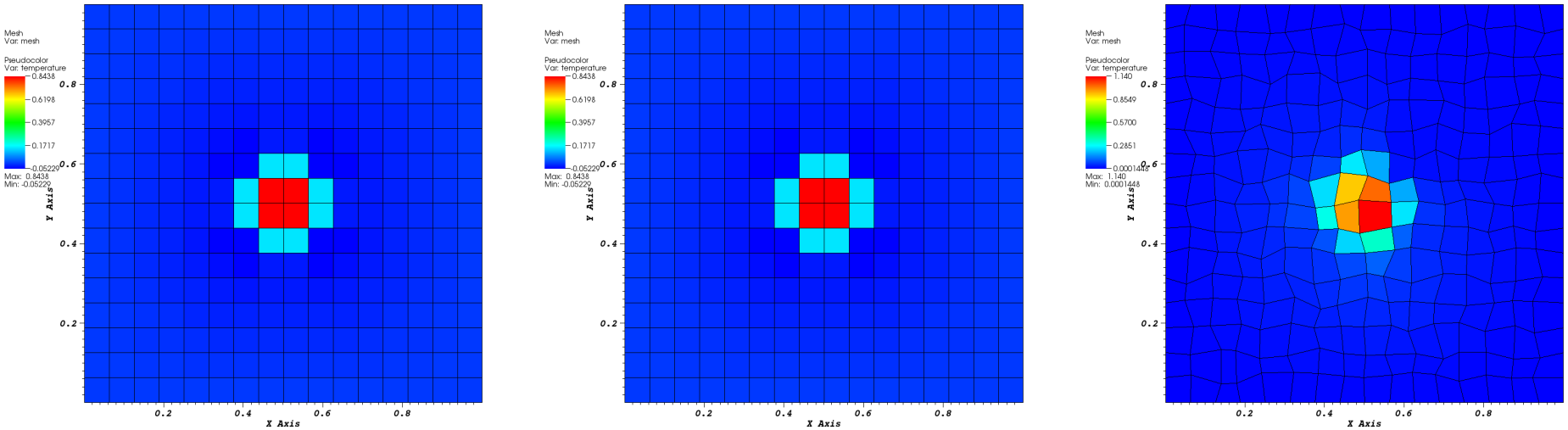
Test case description (Lipnikov 2003)

The solution of this problem is defined as

$$T(x,y) = 1 - \tanh\left(\frac{(x-0.5)^2 + (y-0.5)^2}{0.01}\right). \tag{6}$$

This solution is defined on $[0,1]^2$, where we impose a source term, dirichlet conditions on the boundaries and we use an anisotropic tensor defined as

$$\mathbb{K} = \begin{pmatrix} 4 & 2 \\ -1 & 3 \end{pmatrix}$$



2D AMR non-linear results

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Convergence rate

Uniform mesh						AMR mesh			
# Cells	h	E^h_∞	q^h_∞	E^h_2	q^2_2	# Cells	ratio	E^h_∞	E^h_2
256	6.52D-02	1.10D-01	**2.17**	2.98D-02	**3.28**	256	1.00	1.10D-01	2.98D-02
1 024	3.12D-02	2.44D-02	**2.00**	3.05D-03	**2.04**	556	1.84	2.44D-02	3.05D-03
4 096	1.56D-02	6.09D-03	**2.00**	7.43D-04	**2.01**	988	4.14	6.50D-03	8.02D-04
16 384	7.81D-03	1.52D-03	**1.99**	1.84D-04	**2.00**	3 952	4.14	1.70D-03	2.04D-04
65 536	3.90D-03	3.81D-04	-	4.60D-05	-	15 808	4.14	4.09D-04	5.05D-05

Table : Anisotropic non-linear problem, asymptotic errors in both L^∞ and L^2 norms and corresponding truncation errors using the CCLADNS scheme on a series of uniform and AMR cartesian 2D grids.

3D AMR results

Test case description

The solution of this problem is defined as

$$T(x, y) = 1 - \tanh\left(\frac{(x-0.5)^2 + (y-0.5)^2 + (z-0.5)^2}{0.01}\right). \tag{7}$$

This solution is defined on $[0, 1]^3$, where we impose a source term, dirichlet conditions on the boundaries and we use an anisotropic tensor defined as

$$\mathbb{K} = \begin{pmatrix} 5 & 2 & -1 \\ 3 & 4 & 2 \\ -3 & 0.5 & 6 \end{pmatrix}$$

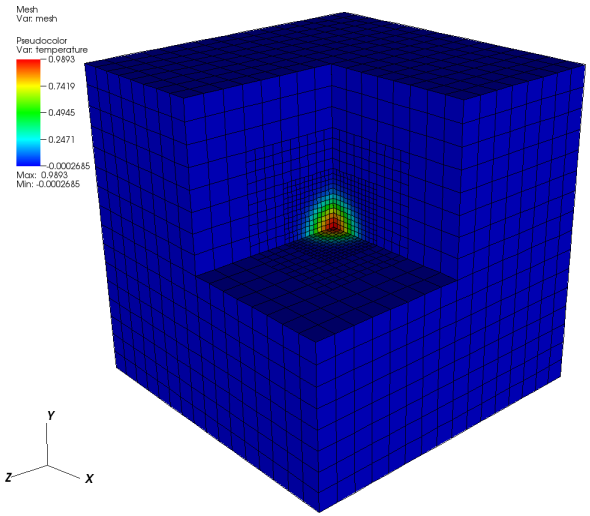

Figure : Solution on 30 304 cells.

Results

Uniform mesh				AMR mesh		
# Cells	h	E_2^h	q_2^2	# Cells	ratio	E_2^h
4 096	6.25D-2	6.68D-03	**2.54**	4 096	1.00	6.68D-03
32 768	3.12D-2	1.14D-03	**2.04**	11 096	2.95	1.14D-03
262 144	1.56D-2	2.78D-04	**2.00**	30 304	8.65	2.78D-04
2 097 152	7.81D-3	6.91D-05	**2.00**	242 432	8.65	6.91D-05
16 777 216	3.90D-3	1.72D-05	-	1 939 456	8.65	1.72D-05

Table : Asymptotic errors in L^2 norms and corresponding truncation errors on a series of uniform and AMR cartesian 3D grids.

Conclusions - Perspectives

Conclusions

- We developped the CCLADNS scheme for solving anisotropic diffusion in 2D and 3D
- This scheme can handle 2D and 3D AMR grids + Hybrid 3D grids (with pyramids)
- Preserves piecewise-linear fields
- Second order convergence rate observed on arbitrary solutions
- It is efficiently parallelized using MPI

Perspectives

- Coupling with Lagrangian Hydrodynamics
- Dynamic AMR
- Dynamic load balancing for parallel computations
- Coupling with 3D ray-tracing for solving laser ablation problems

Article in preparation

A Finite Volume scheme for solving anisotropic diffusion on ALE-AMR grids in three-dimensional geometries. *J. Breil, P. Jacq*

The end

Thank you for your attention!

Sub-cell linear approximation

Let us write the piecewise linear approximation of the temperature over the cell ω_c.

$$T_h(\boldsymbol{x}) = T_c - \mathbb{K}_c^{-1}\boldsymbol{q}_c \cdot (\boldsymbol{x} - \boldsymbol{x}_c). \tag{8}$$

$$\overline{T}_p^f - T_c = -\mathbb{K}_c^{-1}\boldsymbol{q}_c \cdot (\overline{\boldsymbol{x}}_p^f - \boldsymbol{x}_c). \tag{9}$$

Let us note $\boldsymbol{x}_{pc}^f = \overline{\boldsymbol{x}}_p^f - \boldsymbol{x}_c$. Where $\overline{\boldsymbol{x}}_p^f$ denote the vector position of the sub-face centroid. This sub-face centroid can be obtain from the centroid of the two triangles of the sub-face S_{pf}.

$$\overline{T}_p^f - T_c = -\mathbb{K}_c^{-1}\boldsymbol{q}_{pc} \cdot \boldsymbol{x}_{pc}^f. \tag{10}$$

Using the expression of a vector in terms of its normal components

$$\mathbb{J}_{pc}^t \boldsymbol{q}_{pc} = \begin{pmatrix} q_{pc}^1 \\ q_{pc}^2 \\ q_{pc}^3 \end{pmatrix} \tag{11}$$

Where $\mathbb{J}_{pc} = [\boldsymbol{n}_{pc}^1, \boldsymbol{n}_{pc}^2, \boldsymbol{n}_{pc}^3]$. We get

$$\overline{T}_p^f - T_c = -\mathbb{K}_c^{-1}\mathbb{J}_{pc}^{-t} \begin{pmatrix} q_{pc}^1 \\ q_{pc}^2 \\ q_{pc}^3 \end{pmatrix} \cdot \boldsymbol{x}_{pc}^f. \tag{12}$$

$$\boldsymbol{x}_{pc}^f = \mathbb{J}_{pc}^{-t}\mathbb{J}_{pc}^t \boldsymbol{x}_{pc}^f = \mathbb{J}_{pc}^{-t} \begin{pmatrix} \boldsymbol{x}_{pc}^f \cdot \boldsymbol{n}_{pc}^1 \\ \boldsymbol{x}_{pc}^f \cdot \boldsymbol{n}_{pc}^2 \\ \boldsymbol{x}_{pc}^f \cdot \boldsymbol{n}_{pc}^3 \end{pmatrix} \tag{13}$$

Sub-cell linear approximation

We introduce the matrix $\mathbb{X}_{pc}^{i,j} = \boldsymbol{x}_{pc}^{i} \cdot \boldsymbol{n}^{j}$.

$$\overline{T}_p^f - T_c = -\mathbb{J}_{pc}^{-1}\mathbb{K}_c^{-1}\mathbb{J}_{pc}^{-t}\begin{pmatrix} q_{pc}^1 \\ q_{pc}^2 \\ q_{pc}^3 \end{pmatrix} \cdot \begin{pmatrix} \mathbb{x}_{pc}^{f,1} \\ \mathbb{x}_{pc}^{f,2} \\ \mathbb{x}_{pc}^{f,3} \end{pmatrix}. \tag{14}$$

Using all the temperatures of the sub-faces and introducing the sub-cell conductivity tensor

$$\mathbb{K}_{pc} = \mathbb{J}_{pc}^{t}\mathbb{K}_c\mathbb{J}_{pc} \tag{15}$$

$$\begin{pmatrix} \overline{T}_p^1 - T_c \\ \overline{T}_p^2 - T_c \\ \overline{T}_p^3 - T_c \end{pmatrix} = -\mathbb{X}_{pc}\mathbb{K}_{pc}^{-1}\begin{pmatrix} q_{pc}^1 \\ q_{pc}^2 \\ q_{pc}^3 \end{pmatrix}. \tag{16}$$

Which rewrites under the form

$$\begin{pmatrix} q_{pc}^1 \\ q_{pc}^2 \\ q_{pc}^3 \end{pmatrix} = -\mathbb{K}_{pc}\mathbb{X}_{pc}^{-1}\begin{pmatrix} \overline{T}_p^1 - T_c \\ \overline{T}_p^2 - T_c \\ \overline{T}_p^3 - T_c \end{pmatrix}. \tag{17}$$