Mesh regularization for an ALE code based on the limitation of the fluid vorticity

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September, 9th 2015

Motivation

- Material and interface tracking in the context of multiphase flows.
  => Lagrangian formalism is preferred.
- Large deformation are encountered in industrial applications.
  => The mesh may loose some properties: smoothness, convexity of the cells...
- In the indirect ALE formalism (CHLER code, for instance), a remapping phase is needed after the mesh regularization step
- Our work is aimed at introducing an ALE numerical scheme in which the mesh is moving at the same time as the fluid dynamics equations are computed.

This talk is focused on the computation of the velocity of the mesh

1. Methodology
2. Resolution
3. An exact solution to Laplace problem
4. Compressible hydrodynamics test cases

Hodge decomposition

Hodge decomposition of a smooth vector field $u : \Omega \to \mathbb{R}^2$ reads[a]:

$$u = (u_x \, , u_y) = \nabla \, g + \nabla^{\perp} f \, , \tag{1}$$

where

$$\nabla^{\perp} f \equiv \left(-\frac{\partial f}{\partial y} \, , \frac{\partial f}{\partial x} \right) \, , \tag{2}$$

with f et g smooth functions from Ω to $\mathbb{R}$.

If (1) is satisfied, then :

$$\left\{ \begin{array}{ll} \Delta g & = div \; u \\ \Delta f & = rot \; u \end{array} \right. \; , \text{ in } \Omega \tag{3}$$

in 2D, *rot* is defined as:

$$rot \; u = \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \, , \tag{4}$$

[a] It naturally extends to the 3D case.

Existence and uniqueness

Under Poincaré theorem, it is possible to show that there are infinitely many solutions to the previous problem.

For Ω simply connected, the indetermination is removed and it is shown that:
$\exists\,!\,(f,g)\ :\ \Omega\times\Omega\to\mathbb{R}^2\times\mathbb{R}^2$ satisfying the following Laplace equations:

$$\begin{cases} -\Delta f &= -rot\, u\,,\ \text{in } \Omega \\ f &= 0\,,\ \text{on } \partial\Omega\,, \end{cases} \tag{5}$$

and

$$\begin{cases} -\Delta g &= -div\, u\,,\ \text{in } \Omega\,, \\ \frac{\partial g}{\partial n} &= u\cdot n\,,\ \text{on } \partial\Omega\,, \\ \int_\Omega g\, d\, x &= 0\,, \end{cases} \tag{6}$$

In practice, equation 5 is solved, then $\nabla\, g$ is deducted using:

$$\nabla\, g = u - \nabla^{\perp} f$$

Vorticity limitation

Mesh velocity

We propose the following mesh velocity obtained by limiting the vorticity of the Lagrangian velocity

$$w(x) = \nabla g(x) + \theta(x)\, \nabla^{\perp} f(x)\,, \tag{7}$$

with $\theta(x) = \varphi\left(\frac{|\nabla^{\perp} f(x)|}{U_{ref}}\right)$, one can choose: $\varphi_0(r) \equiv \min(1\,, \frac{\chi}{r})\,.$
where U_{ref}, is a reference velocity of the flow and χ the limitation threshold

Comments

At the continuous level:

- If $\forall x, \theta(x) = 1$ => $w = u$: it is a purely Lagrangian method.
- If $\forall x, \theta(x) = 0$ => $w = \nabla g$, vorticity is filtered.

Large Eddy Limitation

- During the time step $\Delta\, t_n$, mesh nodes are moving with velocity w over a distance of the order of $\chi\, U_{ref}\, \Delta\, t_n$,
  with U_{ref} the reference velocity of the flow and χ the limitation threshold.
- This condition on χ is comparable with a Large Eddy Simulation (L.E.S.) philosophy.
- By analogy with the L.E.S. method dedicated to turbulent flows simulation, the L.E.L. strategy is based on the truncation of structures which size is smaller than Δ_{LES}.
  Let us take $\Delta = \chi\, U_{ref}\, \Delta\, t_n$ a reference size of a vortex above which the rotational part of the velocity is limited to compute the evolution of the mesh.

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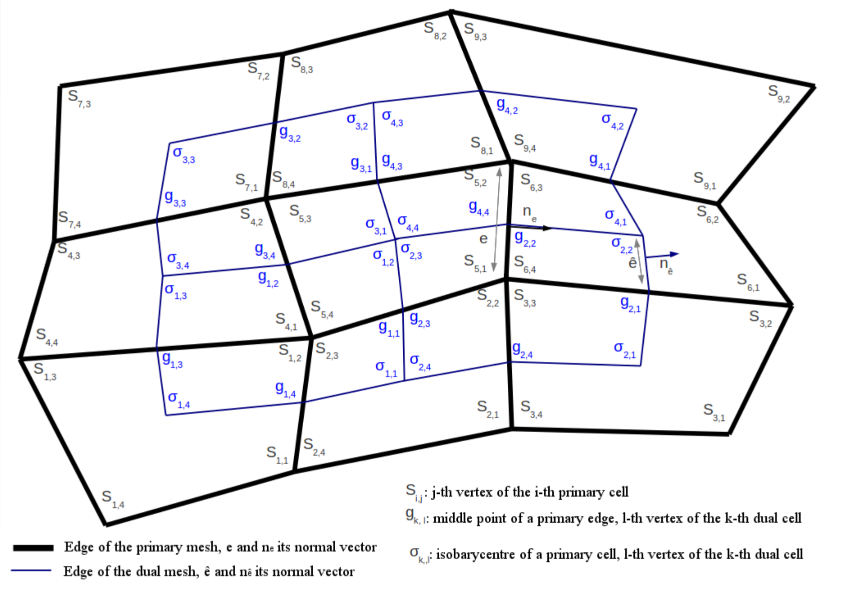

Primary mesh and dual mesh. Notations

Algorithm

- Let us consider a primary mesh and the velocity field associated to its nodes
  1. Determining the isobarycentre in each cell and the midpoint of each edge
     => Dual mesh
  2. Expressing ∇f_g, $\forall g$ midpoint of a primary edge, depending on the unknowns of the problem
  3. Discretizing Laplace equations on each cell (primary and dual)
  4. Writing the linear system
  5. Solving the linear system $\mathbf{AX} = \mathbf{b}$
     => Determining unknown values of f
  6. Computing $\nabla^{\perp} f_S$ for each primal vertex
     => Determining ∇g_S
- The velocity of the mesh is computed using the L.E.L. method.

Linear system layout

Matrix A

$$A = \left(\begin{array}{c|c} A_p & \alpha \\ \hline D & \Delta \end{array}\right)$$

$A_p \in \mathcal{M}_{N_x N_y,(N_x-1)(N_y-1)}(\mathcal{R})$: contribution of primary cells equations on primary vertices.

$\alpha \in \mathcal{M}_{N_x N_y, N_x N_y}(\mathcal{R})$: contribution of primary cells equations on dual vertices.

$D \in \mathcal{M}_{(N_x-1)(N_y-1),(N_x-1)(N_y-1)}(\mathcal{R})$: contribution of dual cells equations on primary vertices.

$\Delta \in \mathcal{M}_{(N_x-1)(N_y-1),N_x N_y}(\mathcal{R})$: contribution of dual cells equations on dual vertices.

Vector of unknowns

$$X = \left(\begin{array}{c} f_{S_1} \\ \vdots \\ f_{S_k} \\ \vdots \\ f_{S_{(N_x-1)(N_y-1)}} \\ \hline f_{\sigma_1} \\ \vdots \\ f_{\sigma_i} \\ \vdots \\ f_{\sigma_{N_x N_y}} \end{array}\right)$$

$f_{S_{1,\ldots,k,\ldots,(N_x-1)(N_y-1)}} \in \mathcal{R}^{(N_x-1)(N_y-1)}$: primary unknowns.

$f_{\sigma_{1,\ldots,i,\ldots,N_x N_y}} \in \mathcal{R}^{N_x N_y}$: dual unknowns.

Right hand side

$$B = \left(\begin{array}{c} b_1 \\ \vdots \\ b_k \\ \vdots \\ b_{N_x N_y} \\ \hline \beta_1 \\ \vdots \\ \beta_i \\ \vdots \\ \beta_{(N_x-1)(N_y-1)} \end{array}\right)$$

$b_{1,\ldots,i,\ldots,N_x N_y} \in \mathcal{R}^{N_x N_y}$: source term on primary cells.

$\beta_{1,\ldots,k,\ldots,(N_x-1)(N_y-1)} \in \mathcal{R}^{(N_x-1)(N_y-1)}$: source term on dual cells.

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Case of a rotational field

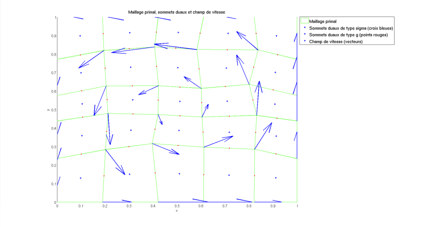
$$\left\{\begin{array}{rcl} -\Delta f & = & -rot\,u\,,\ \text{in}\ \Omega =]0\,,1[^2 \\ f & = & 0\,,\ \text{on}\ \partial\Omega\,, \end{array}\right.$$

Let $u = (u_1\ u_2)$ be the velocity field at the mesh nodes:

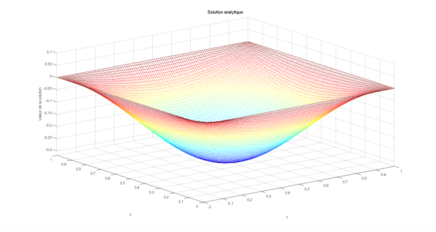
$$\left\{\begin{array}{l} u_1 = sin(\pi\,x)\,cos(\pi\,y) \\ u_2 = -cos(\pi\,x)\,sin(\pi\,y) \end{array}\right. \tag{8}$$

Exact solution :

$$f = \frac{-1}{\pi}\,sin(\pi\,x)\,sin(\pi\,y) \tag{9}$$

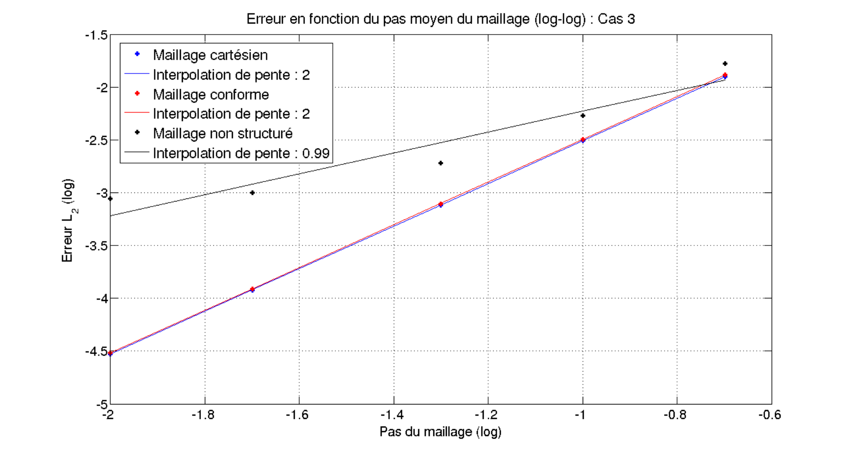


Velocity of the mesh nodes on an unstructured mesh.



Graph of the analytical solution, function f.

Convergence graph



Convergence towards the analytical solution, error in L^2-norm

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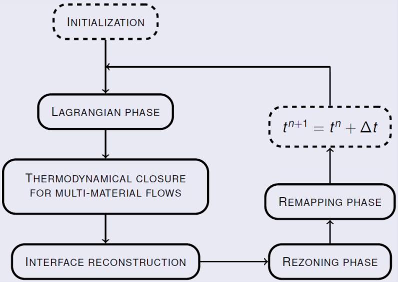
Code CHLER

- CHLER is based on a Lagrangian Finite Volume method dedicated to 2D/3D compressible Euler equations for multi-material flows.
- Polar and Cartesian meshes available.
- Algorithm:

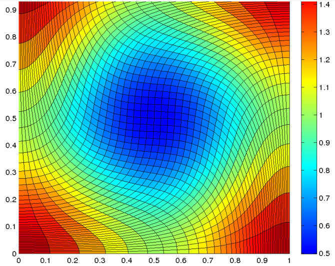


Many options are available:

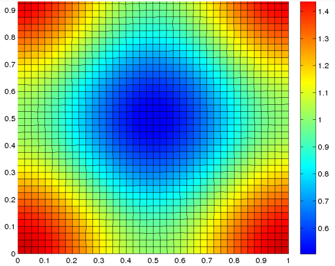
- Lagrangian,
- Lagrange + Remap,
- ALE (various methods are available).

Initial conditions

	Initial state
ρ $[kg.m^{-3}]$	1
p $[Pa]$	$\frac{1}{4}cos(2\pi x)sin(2\pi y)$
u_x $[m.s^{-1}]$	$sin(\pi x)cos(\pi y)$
u_y $[m.s^{-1}]$	$-cos(\pi x)sin(\pi y)$



Lagrangian case



L.E.L. method: rotational part filtered

Taylor-Green vortex 2/2

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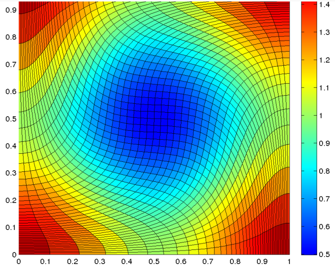
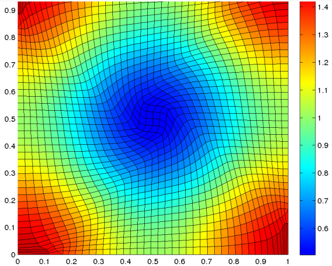


Lagrangian case

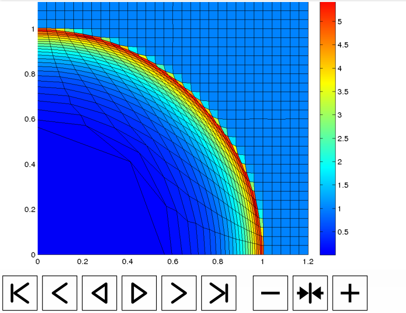


L.E.L. method: rotational part limited

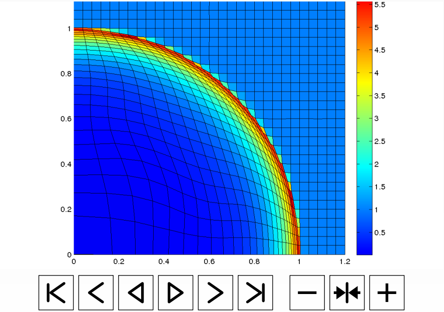
Sedov test-case

Initial conditions

	Central cell	Rest of the domain
$\rho\ [kg.m^{-3}]$	1	1
$p\ [Pa]$	$(\gamma-1)\rho\,\frac{e^0}{Volor}$	10^{-6}
$u_x\ [m.s^{-1}]$	0	0
$u_y\ [m.s^{-1}]$	0	0
γ	1.4	1.4



Lagrangian case



L.E.L. method

Conclusion and future work

Conclusion: L.E.L. method

- Separation of 1D effects (compression, expansion) from 2D effects (rotation) in the velocity field
- Validation and convergence study on exact solutions
- Computation of the mesh velocity by limiting the Lagrangian velocity
- Validation on classical compressible hydrodynamics test cases

Current and future work

- Improve the method to deal with more complex cases (eg. Rayleigh-Taylor instability, Triple point problem)
- Work on the properties of the linear system
- Improving the limitation functions using smooth functions?
- Integration into an ALE hydrodynamics code.