DE LA RECHERCHE À L'INDUSTRIE

An asymptotic preserving cell-centered ALE scheme for a bi-fluid Euler model coupled with friction

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Würzburg, September 2015

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Multi-fluid multi-velocity models with friction appears in

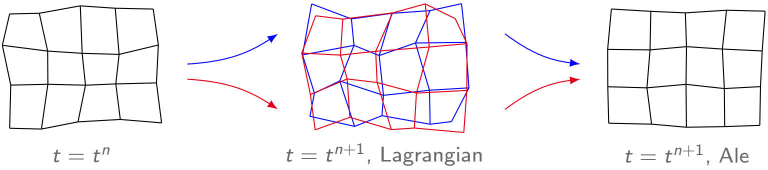
- multiphase flows (eg. [SA99])
- interpenetration mixing models,...

Goal: Scannapieco-Cheng mixing model [SC02, Ena07]

- friction coefficient $\nu := \nu(\delta \mathbf{u}, \rho, ...)$ may vary a lot
- **Need** a scheme that behaves well $\forall \nu \geq 0 \implies$ **Asymptotic Preserving** [Jin99, Gos13, GT02]. (Euler with friction [Fra14])

In this studdy

- ν: positive constant data
- we consider two compressible fluids
- ALE scheme: each fluid has its own grid that must fit at timestep begining



In the following we focus only on the Lagrangian phase

Outline

Bi-fluid model

Lagrangian formulation

Let $\alpha \in \{f_1, f_2\}$ (β denoting the other fluid), the model writes

$$
(1) \qquad
\begin{aligned}
\rho^\alpha D_t^\alpha \tau^\alpha &= \nabla \cdot \mathbf{u}^\alpha, \\
\rho^\alpha D_t^\alpha \mathbf{u}^\alpha &= -\nabla p^\alpha - \nu\rho\delta\mathbf{u}^\alpha, \\
\rho^\alpha D_t^\alpha E^\alpha &= -\nabla \cdot (p^\alpha \mathbf{u}^\alpha) - \nu\rho\delta\mathbf{u}^\alpha \cdot \bar{\mathbf{u}},
\end{aligned}
$$

where
$$
\left|
\begin{array}{ll}
\delta\phi^\alpha = -\delta\phi^\beta = \phi^\alpha - \phi^\beta & \nu\text{: friction} \\
D_t^\alpha := \partial_t + \mathbf{u}^\alpha \cdot \nabla & \implies D_t^\alpha \neq D_t^\beta \\
\rho := \rho^\alpha + \rho^\beta & \rho\bar{\mathbf{u}} := \rho^\alpha \mathbf{u}^\alpha + \rho^\beta \mathbf{u}^\beta
\end{array}
\right.
\qquad p^\alpha = p^\alpha(\rho^\alpha, \epsilon^\alpha)
$$

also, one has

$$
(2) \qquad T^\alpha D_t^\alpha \eta^\alpha \geq \nu \frac{\tau^\alpha}{\tau^\beta} \delta\mathbf{u}^\alpha \cdot \delta\mathbf{u}^\alpha \geq 0.
$$

Conservation

- For each fluid f_1, f_2, the model is conservative in volume and mass
- The model is conservative in the sum of momenta and in the sum of total energies

Limit model

When $\nu \to +\infty$, (1) behaves has the following five equations model. $\forall \alpha \in \{f_1, f_2\}$, β denoting the other fluid

$$\rho D_t \mathbf{u} = -\nabla(p^\alpha + p^\beta),$$
$$\rho^\alpha D_t \tau^\alpha = \nabla \cdot \mathbf{u}, \tag{3}$$
$$\rho^\alpha D_t E^\alpha = -\frac{\rho^\alpha}{\rho}\nabla(p^\alpha + p^\beta) \cdot \mathbf{u} - p^\alpha \nabla \cdot \mathbf{u},$$

Note that $\mathbf{u} = \mathbf{u}^\alpha = \mathbf{u}^\beta$, so $D_t = D_t^\alpha = D_t^\beta$.

Remark

Suming α and β equations gives an Euler mixture model that follows Dalton's law.

Derivation

▸ example

The model is obtained formally by means of Hilbert expension:
Letting $\epsilon = \nu^{-1}$, one writes develops the variables as $\phi = \phi^0 + \epsilon\phi^1 + \mathcal{O}(\epsilon^2)$ and multiplying the obtained equations by powers of ϵ, passes formally to the limit.

Let $\omega \in [0,2]$. Then $\forall \alpha, \beta \in \{f_1, f_2\}$, $\alpha \neq \beta$,

$\rho_r^\alpha := \frac{1}{\#\mathcal{J}_r}\sum_{j\in\mathcal{J}_r}\rho_j^\alpha$, $\quad \rho_r := \rho_r^\alpha + \rho_r^\beta$, $\quad \overline{\mathbf{u}}_r := \frac{\rho_r^\alpha \mathbf{u}_r^\alpha + \rho_r^\beta \mathbf{u}_r^\beta}{\rho_r^\alpha + \rho_r^\beta}$, $\quad \overline{\mathbf{u}}_{jr} := \frac{\rho_r^\alpha \mathbf{u}_j^\alpha + \rho_r^\beta \mathbf{u}_j^\beta}{\rho_r^\alpha + \rho_r^\beta}$.

$$
\begin{aligned}
m_j^\alpha d_t \tau_j^\alpha &= \sum_r \mathbf{C}_{jr}\cdot \mathbf{u}_r^\alpha,\\
d_t m_j^\alpha &= 0,\\
m_j^\alpha d_t \mathbf{u}_j^\alpha &= -\sum_r \mathbf{F}_{jr}^\alpha - \omega \sum_r \nu\rho_r B_{jr}\delta\mathbf{u}_j^\alpha - (1-\omega)\sum_r \nu\rho_r B_{jr}\delta\mathbf{u}_r^\alpha,\\
m_j^\alpha d_t E_j^\alpha &= -\sum_r \mathbf{F}_{jr}^\alpha\cdot\mathbf{u}_r^\alpha - \sum_r \nu\rho_r \overline{\mathbf{u}}_r^T B_{jr}\delta\mathbf{u}_r^\alpha + \omega\sum_r \nu\rho_r \overline{\mathbf{u}}_{jr}^T B_{jr}(\delta\mathbf{u}_r^\alpha - \delta\mathbf{u}_j^\alpha),
\end{aligned}
\tag{4}
$$

where $\mathbf{u}_r^\alpha$ and $\mathbf{F}_{jr}^\alpha$ satisfy

$$
\mathbf{F}_{jr}^\alpha = \mathbf{C}_{jr} p_j^\alpha - A_{jr}^\alpha(\mathbf{u}_r^\alpha - \mathbf{u}_j^\alpha) - \nu\rho_r B_{jr}\delta\mathbf{u}_r^\alpha \quad \text{and} \quad \sum_j \mathbf{F}_{jr}^\alpha = \mathbf{0}. \tag{5}
$$

Blue terms are friction discretization corrections to usual cell-centered schemes.
B_{jr} SPD-matrix such that $\sum_r B_{jr} = V_j I$.

$$
A_{jr} := \overbrace{(\rho c)_j \frac{\mathbf{C}_{jr}\otimes\mathbf{C}_{jr}}{\|\mathbf{C}_{jr}\|}}^{\text{Glace[DM05]}} \quad \text{or} \quad A_{jr} := \overbrace{(\rho c)_j \sum_{i\in\mathcal{F}_{jr}} \frac{\mathbf{N}_{jr}^i\otimes\mathbf{N}_{jr}^i}{\|\mathbf{N}_{jr}^i\|}}^{\text{Eucclhyd [MABO07]}}
$$

$$(5) \implies \underbrace{\sum_j \begin{pmatrix} A_{jr}^{\alpha} + \nu\rho_r B_{jr} & -\nu\rho_r B_{jr} \\ -\nu\rho_r B_{jr} & A_{jr}^{\beta} + \nu\rho_r B_{jr} \end{pmatrix}}_{A_r} \begin{pmatrix} \mathbf{u}_r^{\alpha} \\ \mathbf{u}_r^{\beta} \end{pmatrix} = \underbrace{\sum_j \begin{pmatrix} A_{jr}^{\alpha}\mathbf{u}_j^{\alpha} + \mathbf{C}_{jr} p_j^{\alpha} \\ A_{jr}^{\beta}\mathbf{u}_j^{\beta} + \mathbf{C}_{jr} p_j^{\beta} \end{pmatrix}}_{\mathbf{b}_r}.$$

A_r is a SPD-matrix $\implies \exists!(\mathbf{u}_r^{\alpha}, \mathbf{u}_r^{\beta}) \implies$ the scheme (4)–(5) is well defined.

Property (*a priori* estimate)

Let $(\mathbf{u}_r^{\alpha\,\nu}, \mathbf{u}_r^{\beta\,\nu})$ denote the solution of the linear system for a given ν. One has the following estimates: $\forall\alpha, \beta \in \{f_1, f_2\}, \alpha \neq \beta, \forall\nu \geq 0$

$$^t\mathbf{u}_r^{\alpha\,\nu} A_r^{\alpha} \mathbf{u}_r^{\alpha\,\nu} + {}^t\mathbf{u}_r^{\beta\,\nu} A_r^{\beta} \mathbf{u}_r^{\beta\,\nu} \leq {}^t\mathbf{u}_r^{\alpha\,0} A_r^{\alpha} \mathbf{u}_r^{\alpha\,0} + {}^t\mathbf{u}_r^{\beta\,0} A_r^{\beta} \mathbf{u}_r^{\beta\,0}$$

where $\forall\alpha, A_r^{\alpha} = \sum_j A_{jr}^{\alpha}$, are the nodal matrices of the mono-fluid cell-centered scheme.

Also, one has $\left(\mathbf{u}_r^{\alpha\,\nu} - \mathbf{u}_r^{\beta\,\nu}\right)^T \sum_j B_{jr} \left(\mathbf{u}_r^{\alpha\,0} - \mathbf{u}_r^{\beta\,0}\right) \geq 0$. If $B_{jr} = V_{jr} I$, it implies $\left(\mathbf{u}_r^{\alpha\,\nu} - \mathbf{u}_r^{\beta\,\nu}, \mathbf{u}_r^{\alpha\,0} - \mathbf{u}_r^{\beta\,0}\right) \geq 0$

Property (Conservation)

$\forall \alpha, \beta \in \{f_1, f_2\}, \alpha \neq \beta$, the scheme defined by (4)–(5) ensures conservation

- of mass and volume for each fluid,
- of the sums of the fluids' momenta and total energies.

Property (Entropy)

The first-order continuous in time scheme defined by (4)–(5) satisfies, $\forall \omega \in [0, 2]$, the following entropy inequality $\forall \alpha \in \{f_1, f_2\}$

$$m_j^\alpha T_j^\alpha d_t \eta_j^\alpha \geq \left(1 - \frac{\omega}{2}\right) \sum_r \nu \rho_r^\beta \ {}^t\delta \mathbf{u}_r^\alpha B_{jr} \delta \mathbf{u}_r^\beta + \frac{\omega}{2} \sum_r \nu \rho_r^\beta \ {}^t\delta \mathbf{u}_j^\alpha B_{jr} \delta \mathbf{u}_j^\alpha \geq 0.$$

This inequality is consistent with (2).

Limit scheme

Let $\omega \neq 0$. $\forall \alpha, \beta \in \{f_1, f_2\}, \alpha \neq \beta$, $\forall j \in \mathcal{M}$, if $(\rho_j^\alpha, \mathbf{u}_j^\alpha, E_j^\alpha)$ is constant, then scheme (4)–(5), behaves asymptotically (when $\nu \to +\infty$) as

$$
\begin{aligned}
& m_j^\alpha d_t \tau_j^\alpha = \sum_r \mathbf{C}_{jr} \cdot \mathbf{u}_r, \\
& d_t m_j^\alpha = 0, \\
& (m_j^\alpha + m_j^\beta) d_t \mathbf{u}_j = -\sum_r \mathbf{F}_{jr}^\alpha - \sum_r \mathbf{F}_{jr}^\beta, \\
& m_j^\alpha d_t E_j^\alpha = -\sum_r \mathbf{C}_j p_j^\alpha \cdot \mathbf{u}_r + \sum_r \mathbf{u}_r^T A_{jr}^\alpha (\mathbf{u}_r - \mathbf{u}_j) - \frac{\rho_j^\alpha \rho_j^\beta}{\rho_j} \sum_r \mathbf{u}_j^T \delta \left(\frac{A_{jr}^\alpha}{\rho_j^\alpha} \right) (\mathbf{u}_r - \mathbf{u}_j),
\end{aligned}
\tag{6}
$$

with $\mathbf{F}_{jr}^\alpha + \mathbf{F}_{jr}^\beta = \mathbf{C}_{jr}(p_j^\alpha + p_j^\beta) - (A_{jr}^\alpha + A_{jr}^\beta)(\mathbf{u}_r - \mathbf{u}_j)$, and $\sum_j \mathbf{F}_{jr}^\alpha = \mathbf{0}$.

One has $\mathbf{u}_r = \mathbf{u}_r^\alpha = \mathbf{u}_r^\beta$ and $\mathbf{u}_j = \mathbf{u}_j^\alpha = \mathbf{u}_j^\beta$

Derivation

▸ example

It is obtained formally by means of Hilbert expension.

Property (Consistency)

The limit scheme (6) is weakly consistent with the asymptotic model (3).

▸ sketch of proof

Based on[Des10]. Total energy balance equation is the difficult part.

Asymptotic preserving scheme

In order to studdy the asymptotic preservingness of the scheme, it remains to show that the timestep does not tend to 0 when $\nu \to +\infty$.

Fully discrete scheme (1/5)

Let $\omega \in]0,2]$ and $\theta \in \{n, n+1\}$. Also, $\overline{\mathbf{u}}_{jr}^{\theta} := \frac{\rho_r^{\alpha\,n}\mathbf{u}_j^{\alpha\,\theta}+\rho_r^{\beta\,n}\mathbf{u}_j^{\beta\,\theta}}{\rho_r^{\alpha\,n}+\rho_r^{\beta\,n}}$ and $\overline{\mathbf{u}}_r^{n} = \frac{\rho_r^{\alpha\,n}\mathbf{u}_r^{\alpha\,n}+\rho_r^{\beta\,n}\mathbf{u}_r^{\beta\,n}}{\rho_r^{\alpha\,n}+\rho_r^{\beta\,n}}$.

(7)
$$
\begin{aligned}
\tau_j^{\alpha\,n+1} &= \tau_j^{\alpha\,n} + \frac{\Delta t}{m_j^{\alpha}} \sum_r \mathbf{C}_{jr}^n \cdot \mathbf{u}_r^{\alpha\,n},\\
\mathbf{u}_j^{\alpha\,n+1} &= \mathbf{u}_j^{\alpha\,n} - \frac{\Delta t}{m_j^{\alpha}} \left(\sum_r \mathbf{F}_{jr}^{\alpha,n} + \omega \sum_r \nu \rho_r^n B_{jr}^n \delta \mathbf{u}_j^{\alpha\,\theta} + (1-\omega) \sum_r \nu \rho_r^n B_{jr}^n \delta \mathbf{u}_r^{\alpha\,n} \right),\\
E_j^{\alpha\,n+1} &= E_j^{\alpha\,n} - \frac{\Delta t}{m_j^{\alpha}} \left(\sum_r \mathbf{F}_{jr}^{\alpha,n} \cdot \mathbf{u}_r^{\alpha\,n} + \sum_r \nu \rho_r^n \, {}^t\overline{\mathbf{u}}_r^n B_{jr}^n \delta \mathbf{u}_r^{\alpha\,n} \right.\\
&\qquad \left. - \omega \sum_r \nu \rho_r^n \, {}^t\overline{\mathbf{u}}_{jr}^{\theta} B_{jr}^n \left(\delta \mathbf{u}_r^{\alpha\,n} - \delta \mathbf{u}_j^{\alpha\,\theta} \right) \right),
\end{aligned}
$$

where the $\mathbf{u}_r^{\alpha\,n}$ and $\mathbf{F}_{jr}^{\alpha,n}$ are computed explicitly as

(8)
$$
\mathbf{F}_{jr}^{\alpha,n} = \mathbf{C}_{jr}^n p_j^{\alpha\,n} - A_{jr}^{\alpha,n}(\mathbf{u}_r^{\alpha\,n} - \mathbf{u}_j^{\alpha\,n}) - \nu \rho_r^n B_{jr}^n \delta \mathbf{u}_r^{\alpha\,n}, \quad \text{and} \quad \sum_j \mathbf{F}_{jr}^{\alpha,n} = \mathbf{0}
$$

$$
\theta = \begin{cases} n & \text{explicit scheme,} \\ n+1 & \text{semi-implicit scheme.} \end{cases}
$$

Let $\mathcal{C}^{\alpha\, n} := \left\{ j \in \mathcal{M} / \sum_r \mathbf{C}_{jr}^n \cdot \mathbf{u}_r^{\alpha\, n} < 0 \right\}$, the set of compressive cells.

Property (Positivity of density)

Let us assume that $\forall \alpha \in \{f_1, f_2\}$, $\forall j \in \mathcal{M}$, $\rho_j^{\alpha\, n} > 0$. Let $\Delta t^\rho > 0$ such that,

$$\forall \alpha \in \{f_1, f_2\},\ \forall j \in \mathcal{C}^{\alpha\, n}, \quad \Delta t^\rho < \frac{V_j^{\alpha\, n}}{-\sum_r \mathbf{C}_{jr}^n \cdot \mathbf{u}_r^{\alpha\, n}}.$$

Then, the scheme (7)–(8) defined by $\Delta t = \Delta t^\rho$ ensures, $\forall \omega \in]0, 2]$, that

$$\forall \alpha \in \{f_1, f_2\},\ \forall j \in \mathcal{M}, \quad \rho_j^{\alpha\, n+1} > 0.$$

Property (Positivity of internal energy)

Let us assume that $\forall \alpha \in \{f_1, f_2\}$, $\forall j \in \mathcal{M}$, $e_j^{\alpha\, n} > 0$.
Then, there exists $\Delta t^e > 0$ such that the scheme (7)–(8) defined by $\Delta t = \Delta t^e$ ensures, $\forall \theta \in \{n, n+1\}$ and $\forall \omega \in [0, 2]$, that

$$\forall \alpha \in \{f_1, f_2\},\ \forall j \in \mathcal{M}, \quad e_j^{\alpha\, n+1} > 0.$$

Explicit case $\forall j, \forall \alpha$, $\exists \Delta t_j^{\alpha\, e} > 0$ s.t. (7)–(8) with $\Delta t = \Delta t_j^{\alpha\, e} \implies e_j^{\alpha\, n+1} > 0$.
$\Delta t_j^{\alpha\, e} :=$ positive root of a second-order polynomial (depends on ν).
Δt may tend to 0 when $\nu \to \infty$.

Semi-implicit case In this case, one has to find the **smallest positive** root of a **rationnal function**. However, one can give a sufficient positivity condition:

$$e_j^{\alpha\, n+1} \geq e_j^{\alpha\, n} + \frac{\Delta t}{m_j^\alpha} \left[\sum_r {}^t(\mathbf{u}_j^{\alpha\, n} - \mathbf{u}_r^{\alpha\, n}) A_{jr}^{\alpha\, n} (\mathbf{u}_j^{\alpha\, n} - \mathbf{u}_r^{\alpha\, n}) - \sum_r p_j^{\alpha\, n} \mathbf{C}_{jr}^n \cdot \mathbf{u}_r^{\alpha\, n} \right] - \frac{\Delta t^2}{2 m_j^{\alpha\, 2}} \left(\sum_r A_{jr}^{\alpha\, n} (\mathbf{u}_j^{\alpha\, n} - \mathbf{u}_r^{\alpha\, n}) \right)^2.$$

Right hand side being *algebraically* the internal energy variation for mono-fluid cell-center scheme, *i.e.* it is *algebraically* **independent** of ν.
$\mathbf{u}_r^n = \mathbf{u}_r^n(\nu)$, but $\Delta t^e \underset{\nu \to \infty}{\not\to} 0$ for given $\mathbf{u}_r^n$ to ensure $e_j^{\alpha\, n+1} > 0$

Property (Entropy stability)

Let $U := \left(\tau, \mathbf{u}^T, E\right)^T$ and let η the entropy. There exists $\Delta t^\eta > 0$, such that $\forall \alpha, \beta \in \{f_1, f_2\}$, such that $\alpha \neq \beta$, if the pressure law $p^\alpha : (\rho, e) \to p^\alpha(\rho, e)$ is a differentiable function, then the scheme (7)–(8) defined by $\Delta t = \Delta t^\eta$, $\forall \omega \in]0, 2]$ and $\theta \in \{n, n+1\}$, ensures that,

1. the scheme is entropy stable:
$$\forall j \in \mathcal{M}, \quad \eta\left(U_j^{\alpha\, n+1}\right) \geq \eta\left(U_j^{\alpha\, n}\right),$$
2. and $\forall j \in \mathcal{M}$, one has the following alternative. If $\forall r \in \mathcal{R}_j$, $\mathbf{C}_{jr}^n \cdot \mathbf{u}_r^{\alpha\, n} = \mathbf{C}_{jr}^n \cdot \mathbf{u}_j^{\alpha\, n}$ and $\delta\mathbf{u}_r^{\alpha\, n} - \delta\mathbf{u}_j^{\alpha\, \theta} = \mathbf{0}$, then
$$T_j^{\alpha\, n} m_j^\alpha \frac{\eta(U_j^{\alpha\, n+1}) - \eta(U_j^{\alpha\, n})}{\Delta t} \geq \nu \sum_r \rho_r^\beta \; {}^t\delta\mathbf{u}_r^{\alpha\, n} B_{jr}^n \delta\mathbf{u}_r^{\alpha\, n} + \mathcal{O}(\Delta t),$$
else
$$T_j^{\alpha\, n} m_j^\alpha \frac{\eta(U_j^{\alpha\, n+1}) - \eta(U_j^{\alpha\, n})}{\Delta t} \geq \nu \sum_r \rho_r^\beta \; {}^t\delta\mathbf{u}_r^{\alpha\, n} B_{jr}^n \delta\mathbf{u}_r^{\alpha\, n}.$$

Remark

This is an **existence** *result! It does not provide an explicit $\Delta t^\eta > 0$ for general EOS.*

Property (Entropy stability for ideal gas)

Let $U := \left(\tau, \mathbf{u}^T, E\right)^T$ and let η the entropy. Let f_1 and f_2 be two ideal gases.
Then, one can compute explicitly $\Delta t^\eta > 0$, such that $\forall \alpha, \beta \in \{f_1, f_2\}$, such that $\alpha \neq \beta$, the scheme (7)–(8) defined by $\Delta t = \Delta t^\eta$, $\forall \omega \in]0, 2]$ and $\theta \in \{n, n+1\}$, ensures that, the scheme is entropy stable:

$$\forall j \in \mathcal{M}, \quad \eta\left(U_j^{\alpha\, n+1}\right) \geq \eta\left(U_j^{\alpha\, n}\right).$$

Remark

- *If $\theta = n$, $\Delta t^\eta > 0$ is the positive root of a second-order polynomial that depends on ν. As for internal energy positivity, these term can blow up, analysis not finished.*

  ▸ Δt^η calculation (explicit)

- *If $\theta = n+1$, $\Delta t^\eta > 0$ is the smallest positive root of a rationnal function. For $2 \geq \gamma$, one can show according to the case (compression or expension) that negative termes are bounded independently of ν.*

  ▸ Δt^η calculation (semi-implicit)

  Case $1 < \gamma < 2$ is being analyzed. Not finished yet since very computational, but seems ok.

Reference "naive" scheme

$$\begin{aligned} m_j^\alpha d_t \tau_j^\alpha &= \sum_r \mathbf{C}_{jr} \cdot \mathbf{u}_r^\alpha, \\ d_t m_j^\alpha &= 0, \\ m_j^\alpha d_t \mathbf{u}_j^\alpha &= -\sum_r \mathbf{F}_{jr}^\alpha - \sum_r \nu \rho_r B_{jr} \delta \mathbf{u}_j^\alpha, \\ m_j^\alpha d_t E_j^\alpha &= -\sum_r \mathbf{F}_{jr}^\alpha \cdot \mathbf{u}_r^\alpha - \sum_r \nu \rho_r \overline{\mathbf{u}}_{jr}^T B_{jr} \delta \mathbf{u}_j^\alpha, \end{aligned}$$

where $\mathbf{u}_r^\alpha$ and $\mathbf{F}_{jr}^\alpha$ satisfy

$$\mathbf{F}_{jr}^\alpha = \mathbf{C}_{jr} p_j^\alpha - A_{jr}^\alpha (\mathbf{u}_r^\alpha - \mathbf{u}_j^\alpha) \quad \text{and} \quad \sum_j \mathbf{F}_{jr}^\alpha = \mathbf{0}.$$

Remark

This scheme is conservative, stable and weakly consistent with (1). However, one cannot establish (even formally by means of Hilbert expension) that its limit scheme is consistent with (3). This scheme a priori does not preserve the asymptotic.
It is a good candidate for comparisons.
In order to avoid stability problems, $\delta \mathbf{u}_j^\alpha$ term is implicited in momentum equation.

Data

Ideal gas with $\gamma = 1.4$.
$U := (\rho, \mathbf{u}, p)^T$, $U^L := (1, \mathbf{0}, 1)^T$, $U^R := (0.125, \mathbf{0}, 0.1)^T$, $U^\epsilon := (\epsilon, \mathbf{0}, \epsilon)^T$.
On sets at time $t = 0$

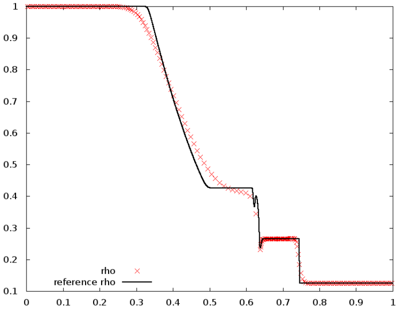
$$U^\alpha(0) = \mathbf{1}_{]0,0.5[}(U^L - U^\epsilon) + \mathbf{1}_{]0.5,1[}U^\epsilon$$
$$U^\beta(0) = \mathbf{1}_{]0,0.5[}U^\epsilon + \mathbf{1}_{]0.5,1[}(U^R - U^\epsilon).$$

In the analyzed scheme, each fluid occupies the whole computational domain.
Reference solution is computed using 10^5 cells.

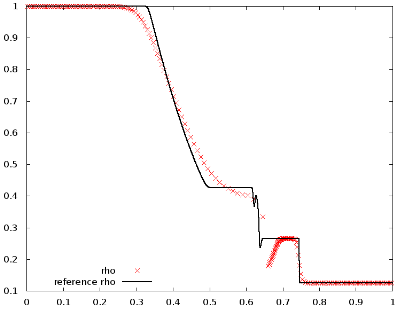
"Sod shock tube", $\nu = 10^2$

Time $t = 0.14$. 200 cells. Fluid α treated as Lagrangian.
Density plot.

AP scheme



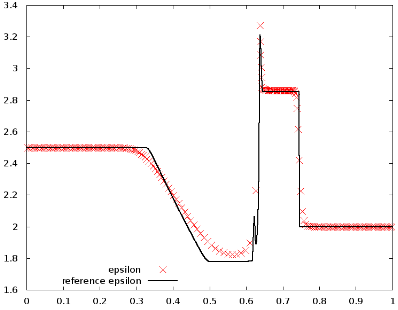
non AP scheme



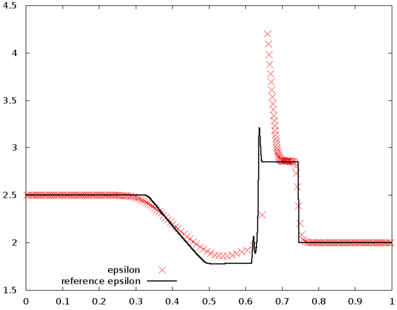
“Sod shock tube”, $\nu = 10^2$

Time $t = 0.14$. 200 cells. Fluid α treated as Lagrangian.
Internal energy plot.

AP scheme

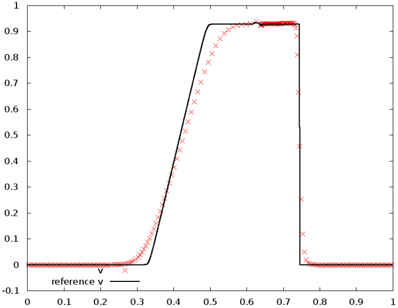


non AP scheme

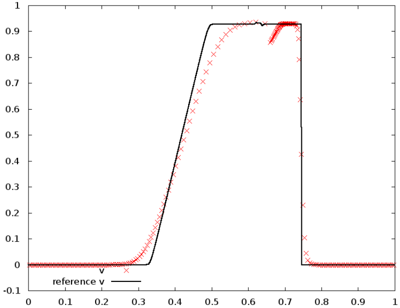


Time $t = 0.14$. 200 cells. Fluid α treated as Lagrangian.
Velocity plot.

AP scheme



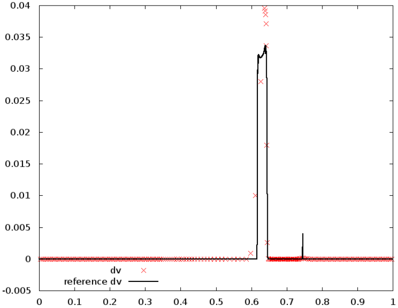
non AP scheme



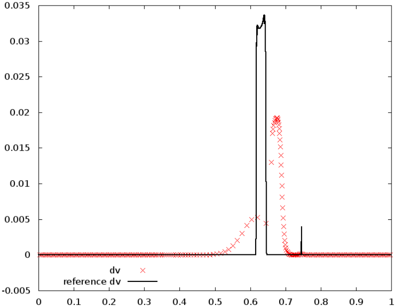
“Sod shock tube”, $\nu = 10^2$

Time $t = 0.14$. 200 cells. Fluid α treated as Lagrangian.
Velocity difference plot.

AP scheme

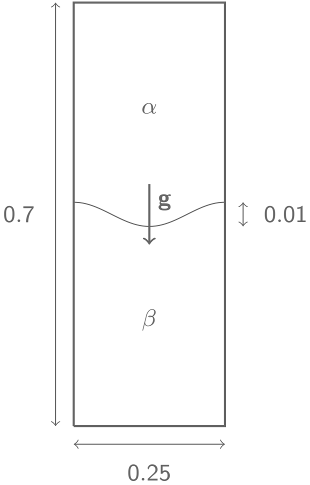


non AP scheme



▸ Results for $\nu = 10^5$.

Rayleigh-Taylor instability



Data

At time $t = 0$, one sets

$$\rho^\alpha = 0.8,$$
$$\rho^\beta = 0.25,$$
$$\mathbf{u}^\alpha = \mathbf{u}^\beta = \mathbf{0},$$
$$p(y) = p_0 + \int_0^y \rho \mathbf{g} \cdot \mathbf{e}_y.$$

Interface position is given by
$f(x) = 0.35 + 0.05\cos(8\pi x)$.

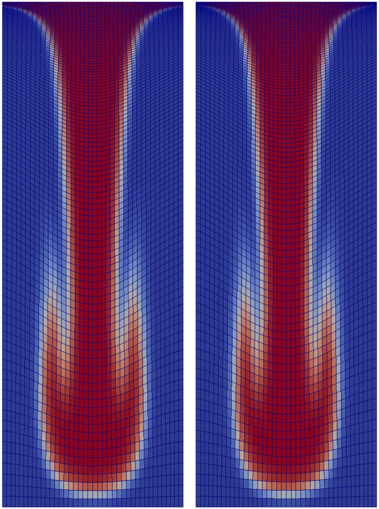
Scheme

A well-balanced gravity discretization is used [CL94].

Tests

- Compares with mono-velocity model,
- ν value variation.

Bi-Fluid model

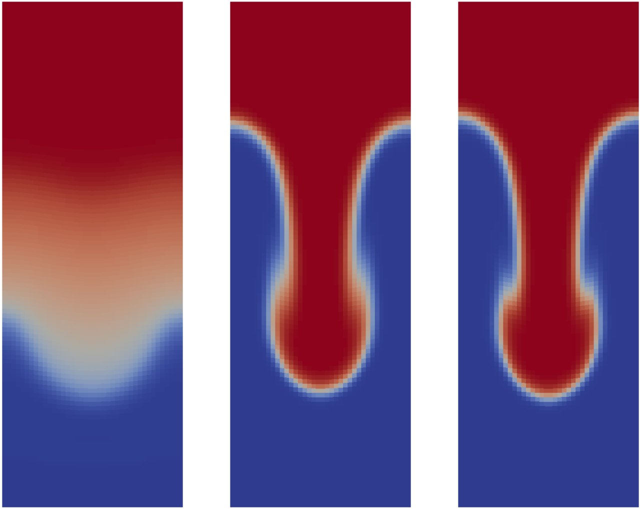


Standard scheme (mono-velocity) +Mixing model

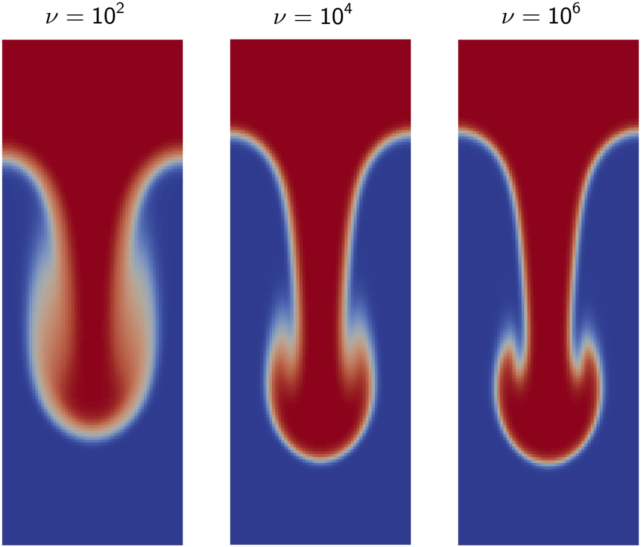
Rayleigh-Taylor instability, ALE simulation, $t = 0.7$

mesh: 40×112

$\nu = 10^2$ $\nu = 10^4$ $\nu = 10^6$



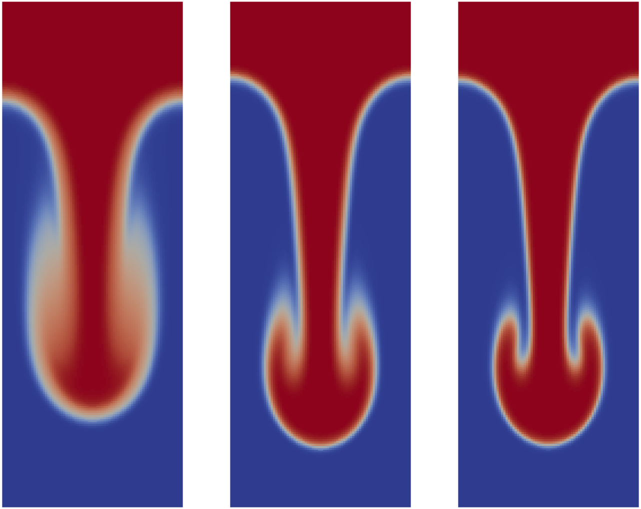
mesh: 60×168

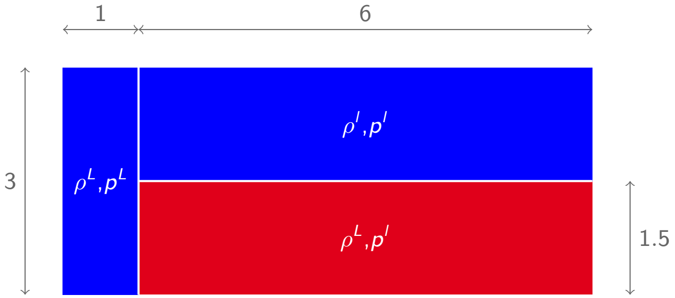


Rayleigh-Taylor instability, ALE simulation, $t = 0.7$

$\nu = 10^2$ $\nu = 10^4$ $\nu = 10^6$

mesh: 80×224



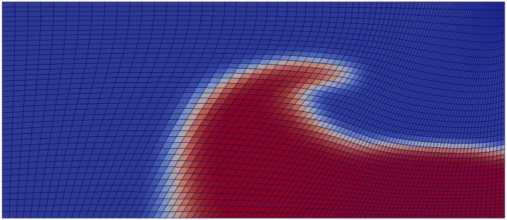


Data

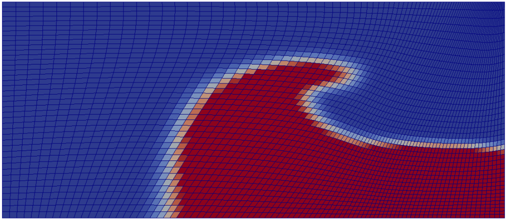
Red fluid is α, blue is β. Initially $\rho^L = 1$, $\rho^l = 0.125$, $p^L = 1$, $p^l = 0.1$, $\mathbf{u} = \mathbf{0}$. $\gamma = 1.4$.

Triple-point problem, $\nu = 10^6$, $t = 5$

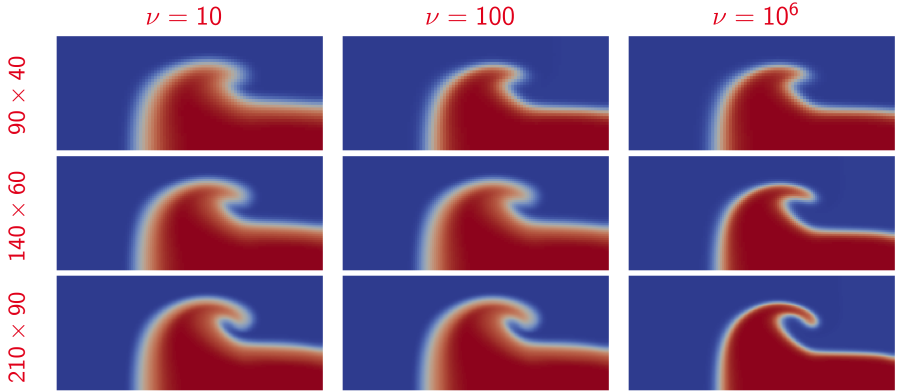
Bi-Fluid model



Standard scheme (mono-velocity)+mixing model



Triple-point problem, $t = 5$



Conclusions

- First-order cell-center scheme for bi-fluid with friction model
  - explicit or sem-implicit treatment of the friction term
  - class of schemes depending on a real parameter ω
- Properties for $\omega \in]0, 2]$
  - conservative
  - stablility
    - density positivity: provided **explicit** $\Delta t > 0$
    - internal energy positivity: provided **explicit** $\Delta t > 0$
    - entropy increase:

      general EOS: **existence** of $\Delta t > 0$
      ideal gas: provided **explicit** $\Delta t > 0$
- Asymptotic preserving
  - limit scheme consistent with limit model
  - $\theta = n + 1$ and $2 \leq \gamma$, timestep does not go to 0 when $\nu \to +\infty$.
- Validated through numerical tests

Perspectives

- $\theta = n + 1$: finish perfect gas stability analysis $1 < \gamma < 2$ (almost done)
- $\theta = n$: can it work? not even tested numerically...
- Varying ν (interpenetration mixing model)
  - analysis should *a priori* be straight forward
  - ω kept uniform or varying with ν?
- second-order (AP analysis?)
- extend to multiple (more than two) fluids
- Differently supported fluids
- Fully Lagrangian approach

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Appendix

Formal calculation of momentum limit equation when $\nu \to +\infty$ in (1)

$$\rho^\alpha D_t^\alpha \mathbf{u}^\alpha = -\nabla p^\alpha - \frac{1}{\epsilon}\rho\delta\mathbf{u}^\alpha, \quad \text{where } \epsilon = \nu^{-1}$$

$$\overset{\rho^\alpha > 0}{\iff} \quad \partial_t \mathbf{u}^\alpha + (\nabla \mathbf{u}^\alpha)\mathbf{u}^\alpha = -\frac{\nabla p^\alpha}{\rho^\alpha} - \frac{1}{\epsilon}\frac{\rho}{\rho^\alpha}\delta\mathbf{u}^\alpha,$$

$$\implies \quad \partial_t(\delta\mathbf{u}^\alpha) + \delta((\nabla\mathbf{u})\mathbf{u})^\alpha = -\delta\left(\frac{\nabla p}{\rho}\right)^\alpha - \frac{1}{\epsilon}\lambda\delta\mathbf{u}^\alpha, \quad \text{where } \lambda = \frac{\rho^2}{\rho^\alpha\rho^\beta}.$$

Hilbert expansion ($\phi = \phi^0 + \epsilon\phi^1 + \mathcal{O}(\epsilon^2)$) for all variables gives

$$(9) \;\; \partial_t(\delta\mathbf{u}^{\alpha,0}) + \delta\left((\nabla\mathbf{u})\mathbf{u}\right)^{\alpha,0} = -\delta\left(\frac{\nabla p}{\rho}\right)^{\alpha,0} - \lambda^0\left(\frac{1}{\epsilon}\delta\mathbf{u}^{\alpha,0} + \delta\mathbf{u}^{\alpha,1}\right) - \lambda^1\delta\mathbf{u}^{\alpha,0} + \mathcal{O}(\epsilon).$$

Formal analysis

- $(9)\times\epsilon \quad \implies \quad \lambda^0\delta\mathbf{u}^{\alpha,0} = \mathcal{O}(\epsilon) \quad \implies \quad \delta\mathbf{u}^{\alpha,0} = \mathbf{0}.$
- $\mathbf{u}^{\alpha,0} = \mathbf{0} \quad \overset{(9)}{\implies} \quad \delta\mathbf{u}^{\alpha,1} = -\frac{1}{\lambda^0}\delta\left(\frac{\nabla p}{\rho}\right)^{\alpha,0}$ and $\mathbf{u}^0 := \overline{\mathbf{u}}^0 = \mathbf{u}^{\alpha,0} = \mathbf{u}^{\beta,0}$ and $D_t := D_t^\alpha = D_t^\beta$.
- So momentum equation reads

$$\rho^{\alpha,0}D_t\mathbf{u}^0 = -\nabla p^{\alpha,0} + \rho^0\frac{1}{\lambda^0}\delta\left(\frac{\nabla p}{\rho}\right)^{\alpha,0} = -\frac{\rho^{\alpha,0}}{\rho^0}\left(p^{\alpha,0} + p^{\beta,0}\right).$$

Property (B. Després [Des10])

$$d_t m_j = 0,$$
$$m_j d_t \tau_j = \sum_r \mathbf{C}_{jr} \cdot \mathbf{u}_r,$$
$$m_j d_t \mathbf{u}_j = -\sum_r \mathbf{F}_{jr},$$
$$m_j d_t E_j = -\sum_r \mathbf{F}_{jr} \cdot \mathbf{u}_r,$$

where $\mathbf{F}_{jr} = \mathbf{C}_{jr} p_j - A_{jr}(\mathbf{u}_r - \mathbf{u}_j),$ and $\sum_j \mathbf{F}_{jr} = \mathbf{0},$

is weakly consistent with the following system of equations

$$\rho D_t \tau = \nabla \cdot \mathbf{u},$$
$$\rho D_t \mathbf{u} = -\nabla p,$$
$$\rho D_t E = -\nabla \cdot p\mathbf{u}.$$

Let $\rho := \rho^\alpha + \rho^\beta$, $E := \frac{\rho^\alpha E^\alpha + \rho^\beta E^\beta}{\rho}$, $c := \frac{\rho^\alpha c^\alpha + \rho^\beta c^\beta}{\rho}$ and $p := p^\alpha + p^\beta$

Property (B. Després [Des10])

$$
\begin{aligned}
d_t m_j &= 0,\\
m_j d_t \tau_j &= \sum_r \mathbf{C}_{jr} \cdot \mathbf{u}_r,\\
m_j d_t \mathbf{u}_j &= -\sum_r \mathbf{F}_{jr},\\
m_j d_t E_j &= -\sum_r \mathbf{F}_{jr} \cdot \mathbf{u}_r,
\end{aligned}
$$

where $\mathbf{F}_{jr} = \mathbf{C}_{jr}(p_j^\alpha + p_j^\beta) - (A_{jr}^\alpha + A_{jr}^\beta)(\mathbf{u}_r - \mathbf{u}_j)$, and $\sum_j \mathbf{F}_{jr} = \mathbf{0}$,

is weakly consistent with the following system of equations

$$
\begin{aligned}
\rho D_t \tau &= \nabla \cdot \mathbf{u},\\
\rho D_t \mathbf{u} &= -\nabla (p^\alpha + p^\beta),\\
\rho D_t E &= -\nabla \cdot (p^\alpha + p^\beta)\mathbf{u}.
\end{aligned}
$$

Proof.

- Consistency for volume, mass and momentum is a direct consequence of [BD]
- Energy balance rewrites:

$$\rho^\alpha D_t E^\alpha = -\nabla\cdot(p^\alpha + p^\beta)\mathbf{u} + p^\beta\nabla\cdot\mathbf{u} + \frac{\rho^\beta}{\rho}\nabla(p^\alpha + p^\beta)\cdot\mathbf{u}.$$

$$\rho_j^\alpha d_t E_j^\alpha = V_j^{-1}\left[-\sum_r \mathbf{C}_j(p_j^\alpha + p_j^\beta)\cdot\mathbf{u}_r + \sum_r \mathbf{u}_r^T(A_{jr}^\alpha + A_{jr}^\beta)(\mathbf{u}_r - \mathbf{u}_j)\right]$$

$$+ V_j^{-1}\left[\sum_r \mathbf{C}_j p_j^\beta\cdot\mathbf{u}_r\right] + V_j^{-1}\left[-\frac{\rho_j^\beta}{\rho_j}\mathbf{u}_j^T\sum_r\left(A_{jr}^\alpha + A_{jr}^\beta\right)(\mathbf{u}_r - \mathbf{u}_j)\right]$$

$$+ V_j^{-1}\left[-\sum_r(\mathbf{u}_r - \mathbf{u}_j)^T A_{jr}^\beta(\mathbf{u}_r - \mathbf{u}_j)\right].$$

Proof.

- Consistency for volume, mass and momentum is a direct consequence of [BD]
- Energy balance rewrites:

$$\rho^\alpha D_t E^\alpha = -\nabla\cdot(p^\alpha + p^\beta)\mathbf{u} + p^\beta \nabla\cdot\mathbf{u} + \frac{\rho^\beta}{\rho}\nabla(p^\alpha + p^\beta)\cdot\mathbf{u}.$$

$$\begin{aligned}
\rho_j^\alpha d_t E_j^\alpha = &\ \overbrace{V_j^{-1}\left[-\sum_r \mathbf{C}_j(p_j^\alpha + p_j^\beta)\cdot\mathbf{u}_r + \sum_r \mathbf{u}_r^T(A_{jr}^\alpha + A_{jr}^\beta)(\mathbf{u}_r - \mathbf{u}_j)\right]}^{\overset{[BD]}{\approx}\left(-\nabla\cdot(p^\alpha+p^\beta)\mathbf{u}\right)\big|_{\mathbf{x}_j}} \\
&+ \underbrace{V_j^{-1}\left[\sum_r \mathbf{C}_j p_j^\beta\cdot\mathbf{u}_r\right]}_{\overset{[BD]}{\approx}\left(p^\beta\nabla\cdot\mathbf{u}\right)\big|_{\mathbf{x}_j}} + \underbrace{V_j^{-1}\left[-\frac{\rho_j^\beta}{\rho_j}\mathbf{u}_j^T\sum_r\left(A_{jr}^\alpha + A_{jr}^\beta\right)(\mathbf{u}_r - \mathbf{u}_j)\right]}_{\overset{[BD]}{\approx}\left(\frac{\rho^\beta}{\rho}\nabla(p^\alpha+p^\beta)\cdot\mathbf{u}\right)\big|_{\mathbf{x}_j}} \\
&+ \underbrace{V_j^{-1}\left[-\sum_r(\mathbf{u}_r - \mathbf{u}_j)^T A_{jr}^\beta(\mathbf{u}_r - \mathbf{u}_j)\right]}_{\to\zeta_j^\alpha\leq 0}.
\end{aligned}$$

Proof.

- Consistency for volume, mass and momentum is a direct consequence of [BD]
- Energy balance rewrites:

$$\rho^\alpha D_t E^\alpha = -\nabla\cdot(p^\alpha + p^\beta)\mathbf{u} + p^\beta \nabla\cdot\mathbf{u} + \frac{\rho^\beta}{\rho}\nabla(p^\alpha + p^\beta)\cdot\mathbf{u}.$$

$$\rho_j^\alpha d_t E_j^\alpha \approx \left(-\nabla\cdot(p^\alpha + p^\beta)\mathbf{u} + p^\beta\nabla\cdot\mathbf{u} + \frac{\rho^\beta}{\rho}\nabla(p^\alpha + p^\beta)\cdot\mathbf{u}\right)\Big|_{\mathbf{x}_j} + \zeta_j^\alpha.$$

$$\rho_j^\alpha d_t E_j^\alpha + \rho_j^\beta d_t E_j^\beta \approx \left(-\nabla\cdot(p^\alpha + p^\beta)\mathbf{u}\right)\Big|_{\mathbf{x}_j} + \zeta_j^\alpha + \zeta_j^\beta,$$

but $\quad \rho_j^\alpha d_t E_j^\alpha + \rho_j^\beta d_t E_j^\beta = \rho_j d_t E_j \overset{[BD]}{\approx} \left(-\nabla\cdot(p^\alpha + p^\beta)\mathbf{u}\right)\Big|_{\mathbf{x}_j}.$

$$\Longrightarrow \underbrace{\zeta_j^\alpha}_{\leq 0} + \underbrace{\zeta_j^\beta}_{\leq 0} = 0 \Longrightarrow \zeta_j^\alpha = \zeta_j^\beta = 0.$$

Ideal gas entropy estimate: explicit case
Case $\gamma \geq 2$

back

$$\frac{m_j^\alpha}{\Delta t}\Delta S \geq (\tau_j^{\alpha\, n})^{\gamma-1}\Bigg[\nu\left(\left(1-\frac{\omega}{2}\right)\sum_r \rho_r^{\beta\, n}\ {}^t\delta\mathbf{u}_r^{\alpha\, n} B_{jr}^n \delta\mathbf{u}_r^{\alpha\, n} + \frac{\omega}{2}\sum_r \rho_r^{\beta\, n}\ {}^t\delta\mathbf{u}_j^{\alpha\, n} B_{jr}^n \delta\mathbf{u}_j^{\alpha\, n}\right)$$

$$+\sum_r {}^t(\mathbf{u}_j^{\alpha\, n}-\mathbf{u}_r^{\alpha\, n})A_{jr}^{\alpha\, n}(\mathbf{u}_j^{\alpha\, n}-\mathbf{u}_r^{\alpha\, n}) + \nu\frac{\omega}{2}\sum_r \rho_r^{\beta\, n}\ {}^t(\delta\mathbf{u}_r^{\alpha\, n}-\delta\mathbf{u}_j^{\alpha\, n})B_{jr}^n(\delta\mathbf{u}_r^{\alpha\, n}-\delta\mathbf{u}_j^{\alpha\, n})\Bigg]$$

$$+\frac{\Delta t}{m_j^\alpha}\Bigg\{-\frac{(\tau_j^{\alpha\, n})^{\gamma-1}}{2}\left(\sum_r A_{jr}^{\alpha\, n}(\mathbf{u}_j^{\alpha\, n}-\mathbf{u}_r^{\alpha\, n}) + \omega\nu\sum_r \rho_r^n B_{jr}^n(\delta\mathbf{u}_j^{\alpha\, n}-\delta\mathbf{u}_r^{\alpha\, n})\right)^2$$

$$-(\gamma-1)(\tau_j^{\alpha\, n})^{\gamma-2}p_j^{\alpha\, n}\left(\sum_r \mathbf{C}_{jr}^n\cdot\mathbf{u}_r^{\alpha\, n}\right)^2$$

$$+(\gamma-1)\left(\sum_r \mathbf{C}_{jr}^n\cdot\mathbf{u}_r^{\alpha\, n}(\tau_j^{\alpha\, n})^{\gamma-2}\right)\Bigg[\left(\sum_r {}^t(\mathbf{u}_j^{\alpha\, n}-\mathbf{u}_r^{\alpha\, n})A_{jr}^{\alpha\, n}(\mathbf{u}_j^{\alpha\, n}-\mathbf{u}_r^{\alpha\, n})\right)$$

$$+\nu\left(\left(1-\frac{\omega}{2}\right)\sum_r \rho_r^{\beta\, n}\ {}^t\delta\mathbf{u}_r^{\alpha\, n} B_{jr}^n \delta\mathbf{u}_r^{\alpha\, n} + \frac{\omega}{2}\sum_r \rho_r^{\beta\, n}\ {}^t\delta\mathbf{u}_j^{\alpha\, n} B_{jr}^n \delta\mathbf{u}_j^{\alpha\, n}\right)$$

$$+\nu\frac{\omega}{2}\sum_r \rho_r^{\beta\, n}\ {}^t(\delta\mathbf{u}_r^{\alpha\, n}-\delta\mathbf{u}_j^{\alpha\, n})B_{jr}^n(\delta\mathbf{u}_r^{\alpha\, n}-\delta\mathbf{u}_j^{\alpha\, n})\Bigg]\Bigg\}$$

$$-\frac{1}{2}\frac{\Delta t^2}{m_j^{\alpha\, 2}}(\gamma-1)\left(\sum_r \mathbf{C}_{jr}^n\cdot\mathbf{u}_r^{\alpha\, n}(\tau_j^{\alpha\, n})^{\gamma-2}\right)\left(\sum_r A_{jr}^{\alpha\, n}(\mathbf{u}_j^{\alpha\, n}-\mathbf{u}_r^{\alpha\, n}) + \omega\nu\sum_r \rho_r^n B_{jr}^n(\delta\mathbf{u}_j^{\alpha\, n}-\delta\mathbf{u}_r^{\alpha\, n})\right)^2.$$

$$
\begin{aligned}
\frac{m_j^\alpha}{\Delta t}\Delta S \geq (\tau_j^{\alpha\,n})^{\gamma-1}\Bigg[\nu\left(\left(1-\frac{\omega}{2}\right)\sum_r \rho_r^{\beta\,n}\ {}^t\delta\mathbf{u}_r^{\alpha\,n}B_{jr}^n\delta\mathbf{u}_r^{\alpha\,n}+\frac{\omega}{2}\sum_r \rho_r^{\beta\,n}\ {}^t\delta\mathbf{u}_j^{\alpha\,n+1}B_{jr}^n\delta\mathbf{u}_j^{\alpha\,n+1}\right) \\
+\sum_r {}^t(\mathbf{u}_j^{\alpha\,n}-\mathbf{u}_r^{\alpha\,n})A_{jr}^{\alpha\,n}(\mathbf{u}_j^{\alpha\,n}-\mathbf{u}_r^{\alpha\,n})+\nu\frac{\omega}{2}\sum_r \rho_r^{\beta\,n}\ {}^t(\delta\mathbf{u}_r^{\alpha\,n}-\delta\mathbf{u}_j^{\alpha\,n+1})B_{jr}^n(\delta\mathbf{u}_r^{\alpha\,n}-\delta\mathbf{u}_j^{\alpha\,n+1})\Bigg] \\
+\frac{\Delta t}{m_j^\alpha}\Bigg\{(\tau_j^{\alpha\,n})^{\gamma-1}\left[-\frac{1}{2}\left(\sum_r A_{jr}^{\alpha\,n}(\mathbf{u}_j^{\alpha\,n}-\mathbf{u}_r^{\alpha\,n})\right)^2+\frac{1}{2}\left(\omega\nu\sum_r \rho_r^n B_{jr}^n(\delta\mathbf{u}_j^{\alpha\,n+1}-\delta\mathbf{u}_r^{\alpha\,n})\right)^2\right] \\
+(\gamma-1)\left(\sum_r \mathbf{C}_{jr}^n\cdot\mathbf{u}_r^{\alpha\,n}(\tau_j^{\alpha\,n})^{\gamma-2}\right)\times\Bigg[\left(\sum_r {}^t(\mathbf{u}_j^{\alpha\,n}-\mathbf{u}_r^{\alpha\,n})A_{jr}^{\alpha\,n}(\mathbf{u}_j^{\alpha\,n}-\mathbf{u}_r^{\alpha\,n})-\sum_r p_j^{\alpha\,n}\mathbf{C}_{jr}^n\cdot\mathbf{u}_r^{\alpha\,n}\right) \\
+\nu\left(\left(1-\frac{\omega}{2}\right)\sum_r \rho_r^{\beta\,n}\ {}^t\delta\mathbf{u}_r^{\alpha\,n}B_{jr}^n\delta\mathbf{u}_r^{\alpha\,n}+\frac{\omega}{2}\sum_r \rho_r^{\beta\,n}\ {}^t\delta\mathbf{u}_j^{\alpha\,n+1}B_{jr}^n\delta\mathbf{u}_j^{\alpha\,n+1}\right) \\
+\nu\frac{\omega}{2}\sum_r \rho_r^{\beta\,n}\ {}^t(\delta\mathbf{u}_r^{\alpha\,n}-\delta\mathbf{u}_j^{\alpha\,n+1})B_{jr}^n(\delta\mathbf{u}_r^{\alpha\,n}-\delta\mathbf{u}_j^{\alpha\,n+1})\Bigg]\Bigg\} \\
+\frac{\Delta t^2}{m_j^{\alpha\,2}}(\gamma-1)\Bigg\{\left(\sum_r \mathbf{C}_{jr}^n\cdot\mathbf{u}_r^{\alpha\,n}(\tau_j^{\alpha\,n})^{\gamma-2}\right)\Bigg[-\frac{1}{2}\left(\sum_r A_{jr}^{\alpha\,n}(\mathbf{u}_j^{\alpha\,n}-\mathbf{u}_r^{\alpha\,n})\right)^2 \\
+\frac{1}{2}\left(\omega\nu\sum_r \rho_r^n B_{jr}^n(\delta\mathbf{u}_j^{\alpha\,n+1}-\delta\mathbf{u}_r^{\alpha\,n})\right)^2\Bigg]\Bigg\}.
\end{aligned}
$$

In order to get the rationnal function, it remains to develop $\delta \mathbf{u}_j^{\alpha\, n+1}$

$$\overbrace{\left(I + \omega\nu\Delta t\left(\frac{1}{m_j^\alpha} + \frac{1}{m_j^\beta}\right)\sum_r \rho_r^n B_{jr}^n\right)}^{C} \delta \mathbf{u}_j^{\alpha\, n+1}$$

$$= \delta \mathbf{u}_j^{\alpha\, n} + \Delta t \sum_r \delta\left(\frac{A_{jr}^n({\mathbf{u}_r}^n - {\mathbf{u}_j}^n)}{m_j}\right)^\alpha$$

$$+ \omega\nu\Delta t\left(\frac{1}{m_j^\alpha} + \frac{1}{m_j^\beta}\right)\sum_r \rho_r^n B_{jr}^n \delta \mathbf{u}_r^{\alpha\, n},$$

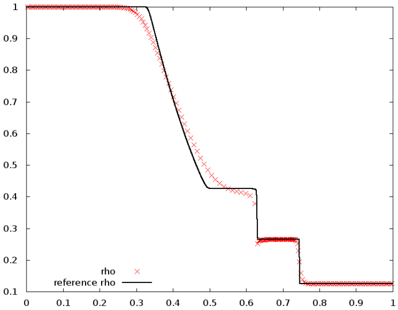
C being a SPD matrix.

“Sod shock tube”, $\nu = 10^5$

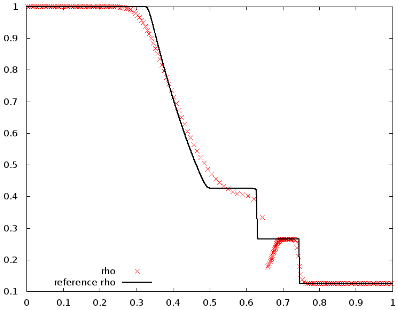
back

Time $t = 0.14$.
Density plot

AP scheme



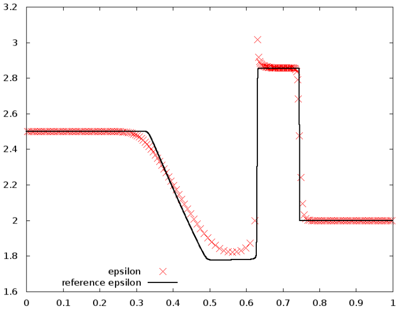
non AP scheme



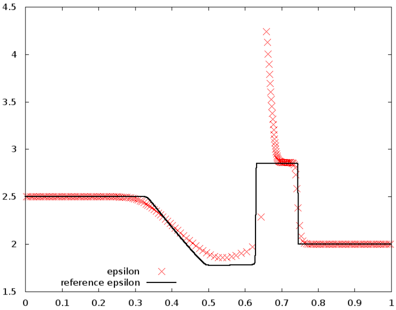
"Sod shock tube", $\nu = 10^5$

Time $t = 0.14$.
Internal energy plot

AP scheme

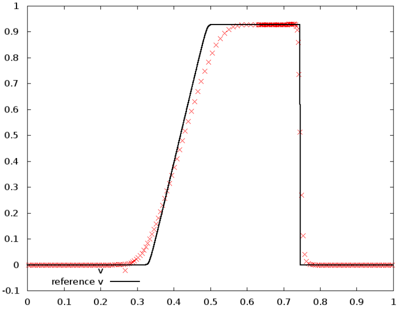


non AP scheme

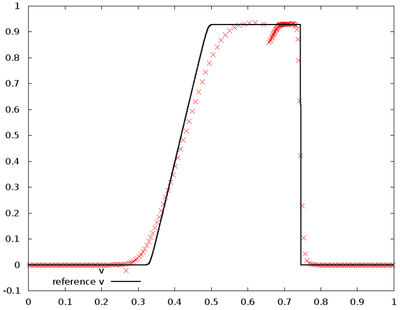


Time $t = 0.14$.
Velocity plot

AP scheme



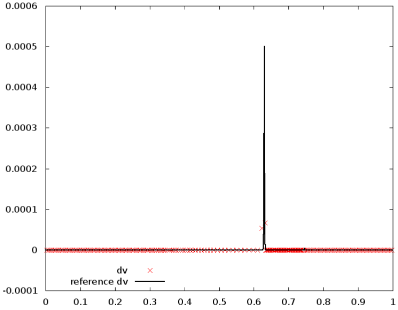
non AP scheme



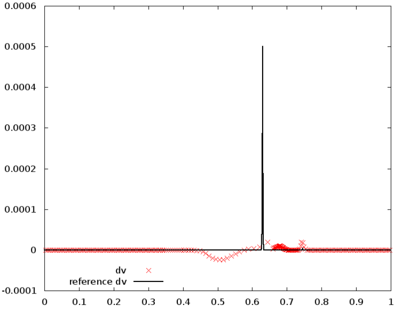
“Sod shock tube”, $\nu = 10^5$

Time $t = 0.14$.
Velocity difference plot

AP scheme



non AP scheme



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