

DE LA RECHERCHE À L'INDUSTRIE



Conservative axisymmetric space- and time-staggered hydrodynamic schemes

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Context

- Notations
 - “VNR” (Von Neumann–Richtmyer, 1D) \equiv “Wilkins” (2D)
 - \equiv “SGH” (Staggered Grid Hydro) \equiv “STS” (Space and Time Staggered)
 - “CSTS” = Conservative Space and Time Staggered
- STS schemes are non-conservative \rightarrow Present trends for Lagrange (and ALE) hydro schemes give preference to **time (and even space) centering**
 - CAVEAT [Addressio et al. 1990]
 - Compatible-Hydro [Caramana et al. 1998]
 - GLACE [Després, Mazeran 2003] [Carré et al. 2009]
 - EUCCLHYD [Maire 2004, 2007]
- STS schemes are **widely used** for their simplicity and robustness
- First works on total energy conservation: [Trulio & Trigger 1960], [Burton 1991]
- Multimat13 & Eccomas14: CSTS Lagrangian scheme in **planar 2D**
- Many studies have been made since [Goad 1960] and [Wilkins 1963] on axisymmetric geometry for space-staggered schemes

Aim of this presentation

- **Axisymmetric** formulation for Conservative Space and Time Staggered Hydrodynamic Schemes
- Different **node masses** definitions
- **Conservation of total energy / spherical symmetry** [Caramana 1998]

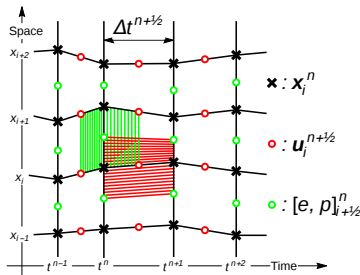
Reminder on CSTS scheme

Lagrangian Conservative Space- and Time- Staggered hydrodynamic scheme
[Claisse, Llor, Fochesato, Multimat13, Eccomas14]

Variant of [Burton 1991]

- same calculation structure as the STS (1, 2, or 3D)
- second order in space and time as the STS
- the volume variations which produce the pressure work are described by the scalar products of corner vectors and displacements: $\frac{\partial V_c}{\partial \mathbf{x}_r} \cdot \mathbf{u}_p$
- exactly conservative in mass, momentum and total energy
- with a novel corrective term in internal energy equation when the time step fluctuates
- with a kinetic energy defined by a positive definite quadratic form of velocity
- second order entropic (predictor–corrector on artificial viscosity)

Variational approach



Consistency demands that **action** \mathcal{A} (= time integral of energy) be built from kinetic and internal energies discretized over the **STS grid**:

$$E_i^n = \sum_c m_c e_c^n$$

$$E_k^{n+1/2} = \sum_p \frac{1}{2} m_p (u_p^{n+1/2})^2$$

$$\mathcal{A} = \sum_n \frac{1}{2} (\Delta t^{n-1/2} + \Delta t^{n+1/2}) E_i^n + \Delta t^{n-1/2} E_k^{n-1/2}$$

- A least action variational principle yields the **only possible** momentum equation: it turns out to be **identical to the original STS scheme**
- Total energy conservation is **deduced** so as to be compatible with the discretization of action

Planar 2D: Momentum equation

obtained by applying a **variational principle to the action integral**

We recover a **second order entropy condition** by applying a **prediction-correction** approach on artificial viscosity q which requires calculating the momentum equation twice:

$$m_p \mathbf{u}_p^{*n+1/2} - m_p \mathbf{u}_p^{n-1/2} = \sum_{c \in C(p)} (p_c^n + q_c^{n-1/2}) \frac{\partial V_c}{\partial \mathbf{x}_p} \Big|_n \frac{\Delta t^{n+1/2} + \Delta t^{n-1/2}}{2}$$

where $q_c^{n-1/2} = \underline{Q}_c(\{\mathbf{u}_p^{n-1/2}\})$ predicted

(1a)

$$m_p \mathbf{u}_p^{n+1/2} - m_p \mathbf{u}_p^{n-1/2} = \sum_{c \in C(p)} (p_c^n + q_c^n) \frac{\partial V_c}{\partial \mathbf{x}_p} \Big|_n \frac{\Delta t^{n+1/2} + \Delta t^{n-1/2}}{2}$$

where $q_c^n = \underline{Q}_c(\{\frac{1}{2}(\mathbf{u}_p^{*n+1/2} + \mathbf{u}_p^{n-1/2})\})$ corrected

(1b)

m_p : constant node mass

Reminder on CSTS scheme

Planar 2D: Internal energy equation

We use the same “energy tally” argument as [Burton 1991]:

- internal energy equation must **match** the kinetic energy equation
→ **only flux terms are left**
- right hand sides of kinetic and internal energies must be opposite up to both **space and time index rearrangements**

$$\begin{aligned}
 m_c(e_c^{n+1} - e_c^n) &= \sum_{p \in P(c)} -\frac{1}{2} \left[(p_c^{n+1} + q_c^{n+1/2}) \frac{\partial V_c}{\partial \mathbf{x}_p} \Big|^{n+1} + (p_c^n + q_c^n) \frac{\partial V_c}{\partial \mathbf{x}_p} \Big|^n \right] \cdot \mathbf{u}_p^{n+1/2} \Delta t^{n+1/2} \\
 &\quad - \frac{1}{2} (q_c^n - q_c^{n-1/2}) \frac{\partial V_c}{\partial \mathbf{x}_p} \Big|^n \cdot \mathbf{u}_p^{n-1/2} \Delta t^{n-1/2} \\
 &\quad + \frac{1}{4} (p_c^n + q_c^n) \frac{\partial V_c}{\partial \mathbf{x}_p} \Big|^n \cdot (\mathbf{u}_p^{n+1/2} - \mathbf{u}_p^{n-1/2}) (\Delta t^{n+1/2} - \Delta t^{n-1/2})
 \end{aligned}$$

Reminder on CSTS scheme

Planar 2D: Internal energy equation

We use the same “energy tally” argument as [Burton 1991]:

- internal energy equation must **match** the kinetic energy equation
→ **only flux terms are left**
- right hand sides of kinetic and internal energies must be opposite up to both **space and time index rearrangements**

$$\begin{aligned}
 & m_c(e_c^{n+1} - e_c^n) \\
 &= \sum_{p \in P(c)} -\frac{1}{2} \left[(p_c^{n+1} + q_c^{n+1/2}) \frac{\partial V_c}{\partial \mathbf{x}_p} \Big|^{n+1} + (p_c^n + q_c^n) \frac{\partial V_c}{\partial \mathbf{x}_p} \Big|^n \right] \cdot \mathbf{u}_p^{n+1/2} \Delta t^{n+1/2} \\
 &\quad - \frac{1}{2} (q_c^n - q_c^{n-1/2}) \frac{\partial V_c}{\partial \mathbf{x}_p} \Big|^n \cdot \mathbf{u}_p^{n-1/2} \Delta t^{n-1/2} \\
 &\quad \text{second order accuracy in time} \\
 &\quad + \frac{1}{4} (p_c^n + q_c^n) \frac{\partial V_c}{\partial \mathbf{x}_p} \Big|^n \cdot (\mathbf{u}_p^{n+1/2} - \mathbf{u}_p^{n-1/2}) (\Delta t^{n+1/2} - \Delta t^{n-1/2}) \\
 &\quad \begin{aligned}
 &- \text{rearrangement of the remaining terms of kinetic energy equation} \\
 &- \text{compatible with causality: no time indices beyond } n+1 \\
 &- \text{small, } \mathcal{O}(\Delta((\Delta t)^2)) \text{ and cancels for constant } \Delta t
 \end{aligned}
 \end{aligned}$$

- Studied options for mass definition
- Wilkins schemes
 - Reminder on classical non-conservative Wilkins STS scheme
 - Conservative approach
- 2D axisymmetric conservative schemes
 - CSTS scheme with variable node masses
 - CSTS scheme with constant coupled node masses
- Summary on schemes' properties
- Comparisons, numerical results

- Studied options for mass definition

- Wilkins schemes

- Reminder on classical non-conservative Wilkins STS scheme
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- 2D axisymmetric conservative schemes

- CSTS scheme with variable node masses
 - CSTS scheme with constant coupled node masses

- Summary on schemes' properties

- Comparisons, numerical results

Studied options for mass definition

- STS_{VWNM} = STS with **Variable** Wilkins Node Masses
- $CSTS_{VWNM}$ = CSTS with **Variable** Wilkins Node Masses
- $CSTS_{VNM}$ = CSTS with **Variable** Node Masses
- $CSTS_{CCNM}$ = CSTS with **Constant** Coupled Node Masses

Studied options for mass definition

- Wilkins schemes

- **Reminder on classical non-conservative Wilkins STS scheme**
Conservative approach

2D axisymmetric conservative schemes

CSTS scheme with variable node masses

CSTS scheme with constant coupled node masses

Summary on schemes' properties

Comparisons, numerical results

Reminder on classical non-conservative Wilkins STS scheme

STS_{VWNM}: STS scheme with **variable** Wilkins node masses definition, [Wilkins 1963]

Momentum equation

- Planar variable node masses

$$m_p^n = \sum_{c \in C(p)} m_{cp}^n, \quad m_{cp}^n = \frac{m_c^n}{|P(c)|} \quad \text{and} \quad m_c^n = \rho_c^n A_c^n$$

- No prediction / correction approach on artificial viscosity

$$m_p^n (\mathbf{u}_p^{n+1/2} - \mathbf{u}_p^{n-1/2}) = \dots$$

Internal energy equation

- Planar formulation for kinetic energy

$$m_p^n [(\mathbf{u}_p^{n+1/2})^2 - (\mathbf{u}_p^{n-1/2})^2] = \dots$$

- No correction to ensure total energy conservation

Properties

- Non-conservative** for momentum equation
- Non-conservative** for total energy
- Entropy production: **order 1** (see A. Marboeuf's poster)
- Spherical symmetry preserved**

Studied options for mass definition

- **Wilkins schemes**

- Reminder on classical non-conservative Wilkins STS scheme

- **Conservative approach**

2D axisymmetric conservative schemes

- CSTS scheme with variable node masses

- CSTS scheme with constant coupled node masses

Summary on schemes' properties

Comparisons, numerical results

Conservative approach

CSTS_{VWNM}: CSTS scheme with **variable** Wilkins node masses

Momentum equation

- Planar variable node masses

$$(m_p^n)_{\text{mom}} = \sum_{c \in C(p)} m_{cp}^n, \quad m_{cp}^n = \frac{m_c^n}{|P(c)|} \quad \text{and} \quad m_c^n = \rho_c^n A_c^n$$

- Prediction / correction approach on artificial viscosity

$$\begin{aligned} (m_p^n)_{\text{mom}} (\mathbf{u}_p^{*n+1/2} - \mathbf{u}_p^{n-1/2}) &= \dots \\ (m_p^n)_{\text{mom}} (\mathbf{u}_p^{n+1/2} - \mathbf{u}_p^{n-1/2}) &= \dots \end{aligned}$$

Kinetic energy equation

- Introduction of variable node masses

$$m_p^n = 2\pi r_p^n (m_p^n)_{\text{mom}}$$

- Conservation

$$m_p^n (\mathbf{u}^{n+1/2})^2 - m_p^{n-1} (\mathbf{u}^{n-1/2})^2 = \dots$$

CSTS_{VWNM}: CSTS scheme with **variable** Wilkins node masses

Internal energy equation

- right hand sides of kinetic and internal energies must be opposite up to both space and time index rearrangements

Properties

- **Non-conservative** for momentum equation
- **Total energy conservation**: by construction
- Entropy production: **order 1 violation** (see A. Marboeuf's poster)
- **Spherical symmetry not preserved**

Studied options for mass definition

Wilkins schemes

Reminder on classical non-conservative Wilkins STS scheme
Conservative approach

- 2D axisymmetric conservative schemes
 - CSTS scheme with variable node masses
 - CSTS scheme with constant coupled node masses

Summary on schemes' properties

Comparisons, numerical results

2D axisymmetric conservative schemes: CSTS_{VNM}

CSTS_{VNM}: CSTS scheme with **variable** node masses (extra forces [Burton 1991])

Momentum equation obtained by applying a variational principle

Prediction-correction approach on artificial viscosity q

$$m_p^n \mathbf{u}_p^{*n+1/2} - m_p^{n-1} \mathbf{u}_p^{n-1/2} = \sum_{c \in C(p)} (p_c^n + q_c^{n-1/2}) \frac{\partial V_c}{\partial \mathbf{x}_p} \Big|_n \frac{\Delta t^{n+1/2} + \Delta t^{n-1/2}}{2} \quad (2a)$$

$$+ \sum_{q \in Q(p)} \frac{1}{2} (\mathbf{u}_q^{n-1/2})^2 \frac{\partial m_q}{\partial \mathbf{x}_p} \Big|_n \Delta t^{n+1/2}$$

where $q_c^{n-1/2} = \underline{Q}[\{\mathbf{u}_p^{n-1/2}\}]$ predicted

$$m_p^n \mathbf{u}_p^{n+1/2} - m_p^{n-1} \mathbf{u}_p^{n-1/2} = \sum_{c \in C(p)} (p_c^n + q_c^n) \frac{\partial V_c}{\partial \mathbf{x}_p} \Big|_n \frac{\Delta t^{n+1/2} + \Delta t^{n-1/2}}{2} \quad (2b)$$

$$+ \sum_{q \in Q(p)} \frac{1}{2} (\mathbf{u}_q^{*n+1/2})^2 \frac{\partial m_q}{\partial \mathbf{x}_p} \Big|_n \Delta t^{n+1/2}$$

where $q_c^n = \underline{Q}[\{\frac{1}{2}(\mathbf{u}_p^{*n+1/2} + \mathbf{u}_p^{n-1/2})\}]$ corrected

Node mass definition

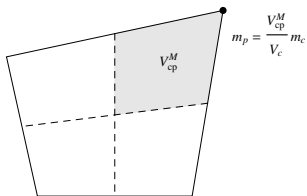
$$m_p^n = m_p [\{\mathbf{x}_p^n\}] = \sum_{c \in C(p)} m_{cp}^n = \sum_{c \in C(p)} m_{cp} [\{\mathbf{x}_p^n\}]$$

$$\text{where } m_{cp}^n = \frac{V_{cp}^n}{V_c^n} m_c$$

V_{cp}^n is the control volume for node p associated to cell c

$$V_{cp} = \frac{2}{3} V_{cp}^M + \frac{1}{3} \left(\frac{1}{4} V_c \right)$$

where V_{cp}^M is the volume delimited by medians [Caramana, Shashkov 1998]

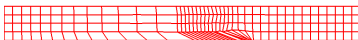


2D axisymmetric conservative schemes: CSTS_{VNM}

Node mass definition

$$V_{cp} = \frac{2}{3} V_{cp}^M + \frac{1}{3} \left(\frac{1}{4} V_c \right)$$

- To avoid polar hollow

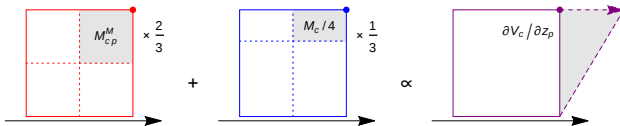


polar hollow with “medians masses”



polar hollow with “Wilkins masses”

- To capture squares on axis $r = 0$



final mesh with “ $\frac{2}{3}$ medians + $\frac{1}{3}$ Wilkins”

- Invariant by affine transforms
- Minimizes spherical symmetry loss
- Other approaches: [Pracht, 75] (no invariance by affine transform), [Goad, 60] (negative for concave cells and create polar hollow)

Kinetic energy equation

Multiplying (2b) by $\frac{1}{2}(\mathbf{u}_p^{n+1/2} + \mathbf{u}_p^{n-1/2})$, we obtain:

$$\begin{aligned} & \frac{1}{2} m_p^n (\mathbf{u}_p^{n+1/2})^2 - \frac{1}{2} m_p^{n-1} (\mathbf{u}_p^{n-1/2})^2 + \frac{1}{2} (m_p^n - m_p^{n-1}) \mathbf{u}_p^{n+1/2} \cdot \mathbf{u}_p^{n-1/2} \\ &= \sum_{c \in C(p)} (p_c^n + q_c^n) \left. \frac{\partial V_c}{\partial \mathbf{x}_p} \right|^n \cdot \frac{\mathbf{u}_p^{n+1/2} + \mathbf{u}_p^{n-1/2}}{2} \frac{\Delta t^{n+1/2} + \Delta t^{n-1/2}}{2} \\ & \quad + \sum_{q \in Q(p)} \frac{1}{2} (\mathbf{u}_q^{*n+1/2})^2 \left. \frac{\partial m_q}{\partial \mathbf{x}_p} \right|^n \cdot \frac{\mathbf{u}_p^{n+1/2} + \mathbf{u}_p^{n-1/2}}{2} \Delta t^{n+1/2} \end{aligned}$$

Internal energy equation

To ensure total energy conservation: right hand sides of kinetic and internal energies must be opposite up to both space and time index rearrangements (take into account **variable node masses terms**)

Properties

- **Momentum conservation:** only along z direction
(dilatation, $\sum_{p \in P(c)} \left| \frac{\partial V_c}{\partial \mathbf{x}_p} \right|^n \cdot \mathbf{e}_r \neq 0$)
- **Total energy conservation:** by construction
- Entropy production: **order 1** (see A. Marboeuf's poster)
- **Spherical symmetry not preserved**

Studied options for mass definition

Wilkins schemes

Reminder on classical non-conservative Wilkins STS scheme

Conservative approach

- 2D axisymmetric conservative schemes

CSTS scheme with variable node masses

- CSTS scheme with constant coupled node masses

Summary on schemes' properties

Comparisons, numerical results

2D axisymmetric conservative schemes: CSTS_{CCNM}

CSTS_{CCNM}: CSTS scheme with **constant** coupled node masses

Momentum equation obtained by applying a variational principle

- Constant node masses

$$\mathbf{M} = (M_{pq})_{p,q}, \quad M_{pq} = \int_{\Omega_0} \rho_0 \Phi_p \Phi_q R dR dZ$$

where Φ_p is p^{th} finite element base function, [Margolin, Nichols 1983], [Dobrev et al. 2010]

- Prediction / correction approach on artificial viscosity

$$\begin{aligned} [\mathbf{M} (\mathbf{u}^{*n+1/2} - \mathbf{u}^{n-1/2})]_p &= \dots \\ [\mathbf{M} (\mathbf{u}^{n+1/2} - \mathbf{u}^{n-1/2})]_p &= \dots \end{aligned}$$

⇒ 2 coupled linear systems to solve

Kinetic energy equation

$$E^{n+1/2} = \sum_p (\mathbf{u}_p^{n+1/2})^t [\mathbf{M} \mathbf{u}^{n+1/2}]_p \rightarrow \text{coupling between neighboring nodes}$$

Internal energy equation

Not modified (version for constant node mass)

Properties

- **Momentum conservation:** only along z direction
(dilatation, $\sum_{p \in P(c)} \left| \frac{\partial V_c}{\partial \mathbf{x}_p} \right|^n \cdot \mathbf{e}_r \neq 0$)
- **Total energy conservation:** by construction
- Entropy production: **order 2** (see A. Marboeuf's poster)
- **Spherical symmetry not preserved**

Studied options for mass definition

Wilkins schemes

Reminder on classical non-conservative Wilkins STS scheme
Conservative approach

2D axisymmetric conservative schemes

CSTS scheme with variable node masses
CSTS scheme with constant coupled node masses

- **Summary on schemes' properties**

Comparisons, numerical results

Summary on schemes' features

	Node mass	Mom. eq.	Kin. eq.
STS planar	$m_p = \sum_c \frac{1}{ P(c) } m_c$	$m_p(u_p^{n+1/2} - u_p^{n-1/2})$	$m_p((u_p^{n+1/2})^2 - (u_p^{n-1/2})^2)$
CSTS planar	$m_p = \sum_c \frac{1}{ P(c) } m_c$	$m_p(u_p^{*n+1/2} - u_p^{n-1/2})$ $m_p(u_p^{n+1/2} - u_p^{n-1/2})$	$m_p((u_p^{n+1/2})^2 - (u_p^{n-1/2})^2)$
STS_{VWNM} axi	$m_p^n = \sum_c \frac{1}{ P(c) } \rho_c^n A_c^n$	$m_p^n(u_p^{n+1/2} - u_p^{n-1/2})$	$m_p^n((u_p^{n+1/2})^2 - (u_p^{n-1/2})^2)$
CSTS_{VWNM} axi	$(m_p^n)_{\text{mom}} = \sum_c \frac{1}{ P(c) } \rho_c^n A_c^n$ $m_p^n = 2\pi r_p^n (m_p^n)_{\text{mom}}$	$(m_p^n)_{\text{mom}}(u_p^{*n+1/2} - u_p^{n-1/2})$ $(m_p^n)_{\text{mom}}(u_p^{n+1/2} - u_p^{n-1/2})$	$m_p^n(u_p^{n+1/2})^2 - m_p^{n-1}(u_p^{n-1/2})^2$
CSTS_{VNM} axi	$m_p^n = \sum_c \frac{V_{cp}^n}{V_c^n} m_c$	$m_p^n u_p^{*n+1/2} - m_p^{n-1} u_p^{n-1/2}$ $m_p^n u_p^{n+1/2} - m_p^{n-1} u_p^{n-1/2}$	$m_p^n(u_p^{n+1/2})^2 - m_p^{n-1}(u_p^{n-1/2})^2$
CSTS_{CCNM} axi	$\mathbf{M} = \int \rho_0 \phi_p \phi_q$	$[\mathbf{M}((u^{*n+1/2} - u^{n-1/2}))]_p$ $[\mathbf{M}((u^{n+1/2} - u^{n-1/2}))]_p$	$(u_p^{n+1/2})^t [\mathbf{M} u^{n+1/2}]_p$

Summary on schemes' properties

	Momentum conservation	Total energy conservation	Node mass definition	Entropy production	Spherical symmetry
STS_{VWNM}	✗	✗	Variable Diagonal Area weighted	Order 1	✓
CSTS_{VWNM}	✗	✓	Variable Diagonal Area weighted	✗	✗
CSTS_{VNM}	only along \mathbf{e}_z	✓	Variable Diagonal Volume weighted	Order 1	✗
CSTS_{CCNM}	only along \mathbf{e}_z	✓	Constant Non-diagonal Volume weighted	Order 2	✗

Studied options for mass definition

Wilkins schemes

- Reminder on classical non-conservative Wilkins STS scheme
- Conservative approach

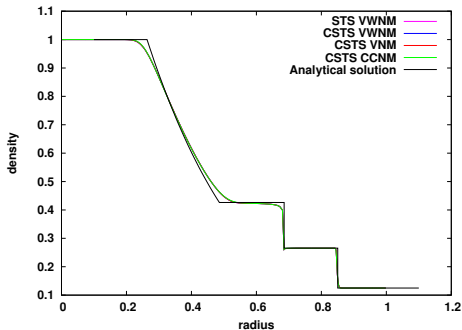
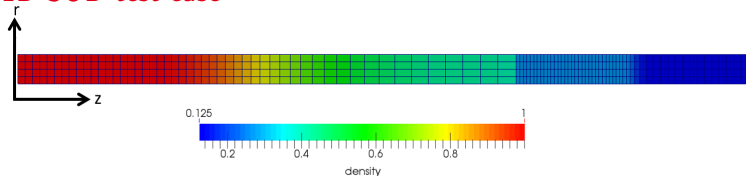
2D axisymmetric conservative schemes

- CSTS scheme with variable node masses
- CSTS scheme with constant coupled node masses

Summary on schemes' properties

- **Comparisons, numerical results**

1D SOD test case



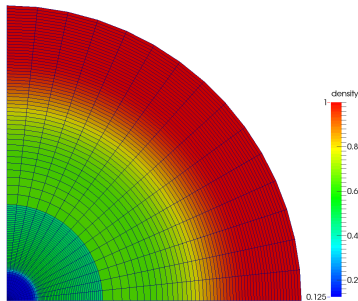
Density profiles

Artificial viscosity coefficients:

$$c_1 = 0.5, c_2 = \frac{\gamma+1}{4}$$

without anti-hourglassing

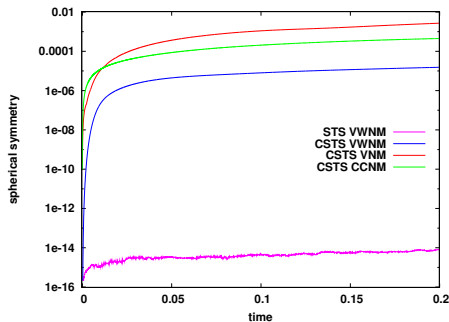
Polar SOD test case



Artificial viscosity coefficients:

$$c_1 = 0.5, c_2 = \frac{\gamma+1}{4}$$

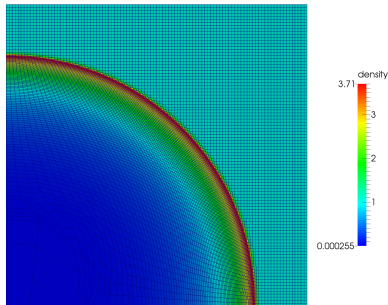
without anti-hourglassing



Spherical symmetry

$$dR = \min_l (\max_{p \in l} R_p - \min_{p \in l} R_p)$$

SEDOV test case

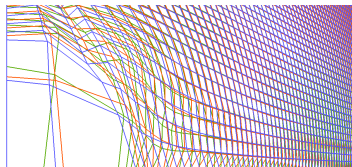


Artificial viscosity coefficients:

$$c_1 = 0.5, c_2 = \frac{\gamma+1}{4}$$

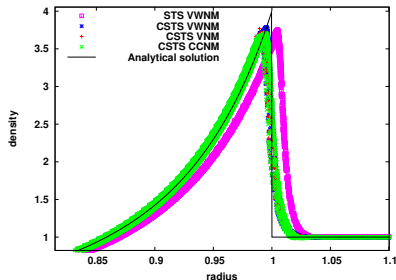
with anti-hourglassing

→ **Better shock capture**
for CSTS schemes
(level and propagation velocity)



Final meshes

$CSTS_{VWNM}$, $CSTS_{VNM}$, $CSTS_{CCNM}$



Density profiles

Conclusion

- Work in progress
- We applied CSTS methodology to axisymmetric geometry: exact energy conservation as expected
- With 4 definitions of node masses (constant, variable, planar, axisymmetric)
- Encouraging first numerical results

Perspectives

- Recover spherical symmetry to scheme's order for CSTS schemes: more complex node mass definition, curvilinear meshes...
- Fully conservative ALE
- Coupling with other physics: elasticity, chemical reaction...