

First-order viscosity for hyperbolic systems

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Acknowledgments

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Richard Pasquetti (Universit de Nice)

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Vladimir Tomov (LLNL)

Young Yang (Texas A&M)

Manual Quezada (Texas A&M)

Support:



2008: 1D Euler flows + Fourier (cannot be higher-order than that!)

- Solution method: Fourier + RK4 + entropy viscosity

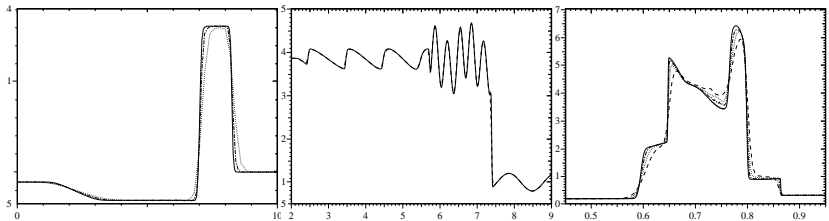
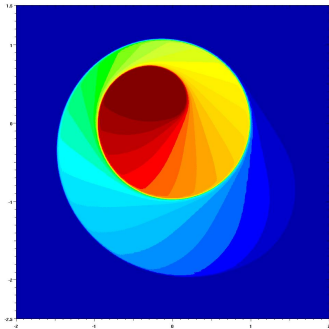
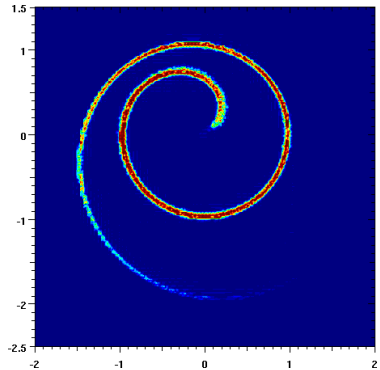


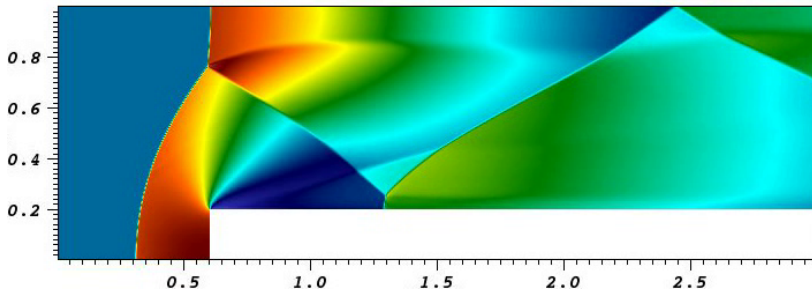
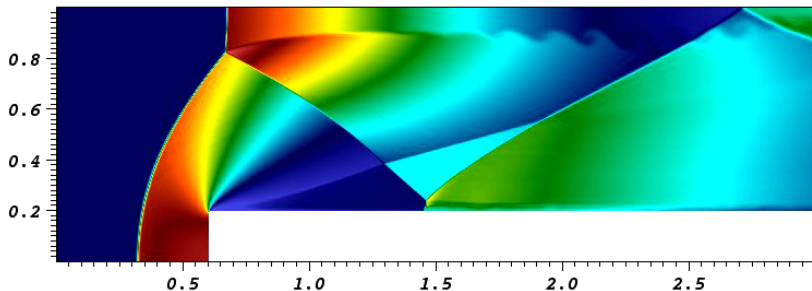
Figure: Lax shock tube, $t = 1.3$, 50, 100, 200 points. Shu-Osher shock tube, $t = 1.8$, 400, 800 points. Right: Woodward-Collera blast wave, $t = 0.038$, 200, 400, 800, 1600 points.



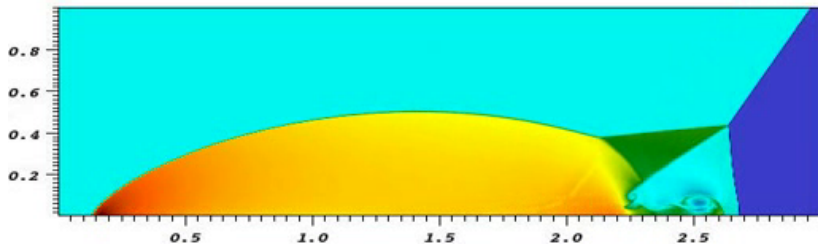
$$\text{KPP: } u_t + (\sin(u))_x + (\cos(u))_y = 0$$

 \mathbb{P}_2 approx \mathbb{Q}_4 entrop viscosity

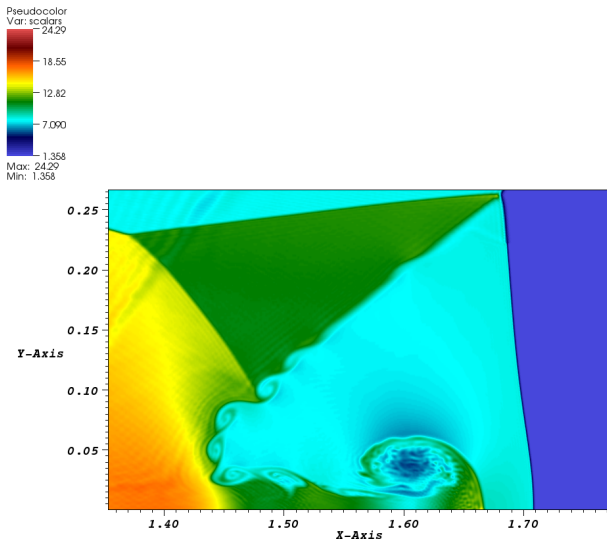
Mach 3 step, ent. vis. sol. $2 \times 10^5 \mathbb{P}_1$ nodes and viscous sol. $3.25 \times 10^5 \mathbb{P}_1$ nodes



Mach 10 Double Mach reflection, ent. vis. sol. $3 \times 10^5 \mathbb{P}_1$ nodes



Mach 10 Double Mach reflection, Zoom



After seven years of EV...



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Soul searching

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- What is an oscillation for a hyperbolic system?



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Outline

1 Back to basics: Scalar conservation



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Back to basics



Introduction

- 1 **Back to basics: Scalar conservation**
- 2 Viscosity for hyperbolic systems



Back to basics: Scalar conservation

Scalar conservation

- Scalar conservation equation (u dependent variable)

$$\begin{aligned}\partial_t u + \nabla \cdot \mathbf{f}(u) &= 0, & (\mathbf{x}, t) &\in \Omega \times \mathbb{R}_+. \\ u(\mathbf{x}, 0) &= u_0(\mathbf{x}), & \mathbf{x} &\in \Omega.\end{aligned}$$



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- Ω open polyhedral domain in \mathbb{R}^d .
- $\mathbf{f} \in \text{Lip}(\mathbb{R}; \mathbb{R}^d)$, the flux.
- $u_0 \in L^\infty(\Omega)$, initial data.
- Periodic BCs or u_0 has compact support (to simplify BCs).



Formulation of the problem (scalar conservation)

Proposition (Entropy condition)

- *The problem has a unique entropy solution $u(x, t)$ such that*

$$\partial_t E(u) + \nabla \cdot \mathbf{F}(u) \leq 0$$

*for all convex entropy $E \in C^2(\mathbb{R}; \mathbb{R})$ and associated entropy flux $\mathbf{F} \in C^2(\mathbb{R}; \mathbb{R})$.
Kruskov (1970) and **Bardos-LeRoux-Nédélec (1979)**.*



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- The entropy solution satisfies the maximum principle

$$u_{0 \min} := \inf_{\xi \in \Omega} u_0(\xi) \leq u(x, t) \leq \max_{\xi \in \Omega} u_0(\xi) := u_{0 \max}, \quad \text{a.e. } x \in \Omega, t \in \mathbb{R}_+.$$



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Definition/Proposition (Invariant domain)

The interval $[u_{0 \min}, u_{0 \max}]$ is an **invariant domain** for the entropy solution;

$$u(\cdot, t) \in [u_{0 \min}, u_{0 \max}], \quad \forall t \geq 0.$$



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The first order viscosity

- Let K be a cell in \mathcal{K} . Let n_K be the number of vertices in K .



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- Define

$$b_K(\varphi_j, \varphi_i) = \begin{cases} -\frac{1}{n_K-1} |K| & \text{if } i \neq j, \quad i, j \in \mathcal{I}(K), \\ |K| & \text{if } i = j, \quad i, j \in \mathcal{I}(K), \\ 0 & \text{if } i \notin \mathcal{I}(K) \text{ or } j \notin \mathcal{I}(K). \end{cases}$$



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- Define ν_K

$$\nu_K^{V,k} = \max_{i \neq j \in \mathcal{I}(K)} \frac{\max(0, \int_{S_{ij}} (\mathbf{f}'(u_h^k) \cdot \nabla \varphi_j) \varphi_i \, dx)}{-\sum_{T \subset S_{ij}} b_T(\varphi_j, \varphi_i)}.$$



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- No restriction on the mesh geometry.
- Observe the automatic rescaling of ν_K^V (i.e. global scaling of b_K does not matter).
- Define entropy viscosity

$$\nu_K^{E,k} := \min(\nu_K^{V,k}, \frac{R_K^k}{\|E(u_h^k) - \bar{E}(u_h^k)\|_{L^\infty(\Omega)}})$$

$$R_K^k(u_h) = \|\frac{1}{\Delta t^{k-1}}(E(u_h^k) - E(u_h^{k-1})) + \mathbf{f}'(u_h^k) \cdot \nabla E(u_h^k)\|_{L^\infty(K)}.$$



The scheme

The scheme

- Advance in time as follows:

$$U_i^{k+1} = U_i^k - \Delta t^k m_i^{-1} \sum_{K \subset S_i} \left(\nu_K^k b_K(u_h^k, \varphi_i) + \int_K \nabla \cdot (\mathbf{f}(u_h^k)) \varphi_i \, d\mathbf{x} \right).$$



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- $\nu_K^k = \nu_K^{V,k}$ gives an invariant domain preserving scheme.
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- $\nu_K^k = \nu_K^{V,k}$ gives an invariant domain preserving scheme.
- $\nu_K^k = \nu_K^{E,k}$ does not give such a scheme.
- FCT between the viscosity and the entropy viscosity methods gives an invariant domain preserving scheme.



Systems



Hyperbolic systems

- 1 Back to basics: Scalar conservation
- 2 **Viscosity for hyperbolic systems**



Hyperbolic systems

The PDEs

- Hyperbolic system

$$\begin{aligned}\partial_t \mathbf{U} + \nabla \cdot \mathbf{F}(\mathbf{U}) &= 0, & (\mathbf{x}, t) &\in \Omega \times \mathbb{R}_+. \\ u(\mathbf{x}, 0) &= \mathbf{U}_0(\mathbf{x}), & \mathbf{x} &\in \Omega.\end{aligned}$$



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Assumptions

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- The problem has a unique “entropy” solution of the 1D Riemann problem.
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- $S(\mathbf{U}_0)$ is convex.



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Examples

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- Euler: $S(\mathbf{U}_0) = \{\rho > 0, e > 0, s \geq s_0\}$.



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Examples

- Invariant domains are convex for genuinely nonlinear systems (**Hoff (1985)**, **Chueh, Conley, Smoller (1973)**).
- Euler: $S(\mathbf{U}_0) = \{\rho > 0, e > 0, s \geq s_0\}$.
- p-system (1D): $\mathbf{U} = (v, u)^T$

$$S(\mathbf{U}_0) := \{\mathbf{U} \in \mathbb{R}_+ \times \mathbb{R}; \inf_{\mathbf{U} \in \mathbf{U}_0} W_2(\mathbf{U}) \leq W_2(\mathbf{U}), W_1(\mathbf{U}) \leq \sup_{\mathbf{U} \in \mathbf{U}_0} W_1(\mathbf{U})\}.$$

where W_1 and W_2 are the Riemann invariants

$$W_1(\mathbf{U}) = u + \int_v^\infty \sqrt{-p'(s)} ds, \quad \text{and} \quad W_2(\mathbf{U}) = u - \int_v^\infty \sqrt{-p'(s)} ds.$$



Approximation (time and space)

Objectives

- Approximate conservation equation in time and space.



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- Use C^0 finite elements.



Approximation (time and space)

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- Use \mathcal{C}^0 finite elements.
- Satisfy the invariant domain property at every time step



Approximation (time and space)

Algorithm: Galerkin + First-order viscosity + Explicit Euler

- $\{\varphi_1, \dots, \varphi_N\}$ nodal Lagrange basis

$$m_i \frac{\mathbf{U}_i^{n+1} - \mathbf{U}_i^n}{\Delta t} + \int_{\Omega} \nabla \cdot (\pi_h(\mathbf{F}(\mathbf{U}^n))) \varphi_i \, dx + \sum_{K \in \mathcal{K}_h} \nu_K b_K(\mathbf{U}^n, \varphi_i) = 0.$$



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- Introduce

$$\mathbf{C}_{ij} = - \int_{\Omega} \nabla \varphi_j \varphi_i \, d\mathbf{x}, \quad D_{ij} = - \sum_K \nu_K b_K(\varphi_i, \varphi_j).$$



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- Then

$$m_i \frac{\mathbf{U}_i^{n+1} - \mathbf{U}_i^n}{\Delta t} = \sum_j (\mathbf{C}_{ij} \cdot \mathbf{F}(\mathbf{U}_j) + D_{ij} \mathbf{U}_j).$$



Approximation (time and space)

Algorithm: Galerkin + First-order viscosity + Explicit Euler

- Observe that conservation implies $\sum_j \mathbf{C}_{ij} = 0$ and $\sum_j D_{ij} = 0$.

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- Try to construct convex combination ...

$$\begin{aligned} \mathbf{U}_i^{n+1} &= \mathbf{U}_i^n \left(1 + 2 \frac{\Delta t}{m_i} D_{ii}\right) + \sum_{j \neq i} \frac{\Delta t}{m_i} (\mathbf{C}_{ij} \cdot (\mathbf{F}(\mathbf{U}_j) - \mathbf{F}(\mathbf{U}_i)) + D_{ij}(\mathbf{U}_j + \mathbf{U}_i)) \\ &= \mathbf{U}_i^n \left(1 - \sum_{j \neq i} 2 \frac{\Delta t}{m_i} D_{ij}\right) + \sum_{j \neq i} \frac{2\Delta t}{m_i} D_{ij} \left(\frac{\mathbf{C}_{ij}}{2D_{ij}} \cdot (\mathbf{F}(\mathbf{U}_j) - \mathbf{F}(\mathbf{U}_i)) + \frac{1}{2}(\mathbf{U}_j + \mathbf{U}_i) \right) \end{aligned}$$



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- Introduce intermediate states $\bar{\mathbf{U}}(\mathbf{U}_i, \mathbf{U}_j)$

$$\bar{\mathbf{U}}(\mathbf{U}_i, \mathbf{U}_j) := \frac{1}{2}(\mathbf{U}_j + \mathbf{U}_i) + \frac{\mathbf{C}_{ij}}{2D_{ij}} \cdot (\mathbf{F}(\mathbf{U}_j) - \mathbf{F}(\mathbf{U}_i)).$$



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- Are the states $\bar{\mathbf{u}}(\mathbf{u}_i, \mathbf{u}_j)$ good objects?



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- Define $\mathbf{n}_{ij} = \mathbf{C}_{ij} / \|\mathbf{C}_{ij}\|_{\ell^2} \in \mathbb{R}^d$, (unit vector).



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provided

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β_{ij} is max wave speed for Riemann problem (easy to find an upper bound).

This becomes the definition of viscosity coefficient $D_{ij} \geq 0$, thereby defining ν_K

$$\nu_K := \max_{i,j \in \mathcal{I}(K)} \frac{\beta_{ij} \|\mathbf{C}_{ij}\|_{\ell^2}}{\sum_{K \in S_{ij}} |b_K(\varphi_j, \varphi_i)|}.$$



Approximation (time and space)

Algorithm: Galerkin + First-order viscosity + Explicit Euler

- CFL condition implies by

$$(1 - \sum_{j \neq i} 2 \frac{\Delta t}{m_i} D_{ij}) \geq 0.$$

- Let $S(\{\mathbf{U}_i, \mathbf{U}_j\})$ be any invariant domain of the Riemann problem.



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Theorem

Provided CFL condition,

- *Local invariance:* $\mathbf{U}_i^{n+1} \in \text{Conv}(\mathbf{U}_i^n, S(\{\mathbf{U}_i, \mathbf{U}_j\}_{i \neq j \in \mathcal{I}(S_i)}))$.
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How to find the local viscosity?

Lemma

$\bar{\mathbf{U}}(\mathbf{U}_i, \mathbf{U}_j)$ is average of *exact solution of fake 1D Riemann problem!*

$$\bar{\mathbf{U}}(\mathbf{U}_i, \mathbf{U}_j) := \frac{1}{2}(\mathbf{U}_j + \mathbf{U}_i) + \frac{\|\mathbf{C}_{ij}\|_{\ell^2}}{2D_{ij}}(\mathbf{f}_{ij}(\mathbf{U}_j) - \mathbf{f}_{ij}(\mathbf{U}_i)).$$

provided

$$\frac{\|\mathbf{C}_{ij}\|_{\ell^2}}{2D_{ij}}\beta_{ij} \leq \frac{1}{2},$$

β_{ij} is max wave speed for Riemann problem.

$$\nu_K := \max_{i,j \in \mathcal{I}(K)} \frac{\beta_{ij} \|\mathbf{C}_{ij}\|_{\ell^2}}{\sum_{K \in S_{ij}} |b_K(\varphi_j, \varphi_i)|}.$$



1D Riemann solver: maximum speed bound

Euler system: $p = (\gamma - 1)\rho e$

Given the states $U_L : U_j$ and $U_R := U_i$, we have

$$\lambda_1 = u_L - a_L \left(1 + \frac{(p^* - p_L)_+}{p_L} \frac{\gamma + 1}{2\gamma} \right)^{\frac{1}{2}} < \lambda_3 = u_R + a_R \left(1 + \frac{(p^* - p_R)_+}{p_R} \frac{\gamma + 1}{2\gamma} \right)^{\frac{1}{2}}$$

and define

$$\beta_{LR} = \max(|\lambda_1|, |\lambda_3|).$$



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In practice we need a good upper bound of p^* : $\bar{p}^* \geq p^*$. Then

$$\beta_{LR} = \max(|\bar{\lambda}_1|, |\bar{\lambda}_3|).$$



Comments, work in progress, future work

Continuous finite elements

- Continuous FE are viable tools to solve hyperbolic systems.
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- Extension to Lagrangian hydrodynamics, DG, other systems.



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Main questions

- How to compute the viscosity coefficient fast?
- What to limit for systems to get invariant domain property?



Thank you

