

# Optimal Control and Parabolic Reinitialization for Level Set Methods

## Level Set Method

- Implicit description of evolving interfaces

$$\Gamma(t) = \Gamma(\varphi(\mathbf{x}, t)) = \{\mathbf{x} \in \Omega \mid \varphi(\mathbf{x}, t) = 0\}$$

- Level set transport equation

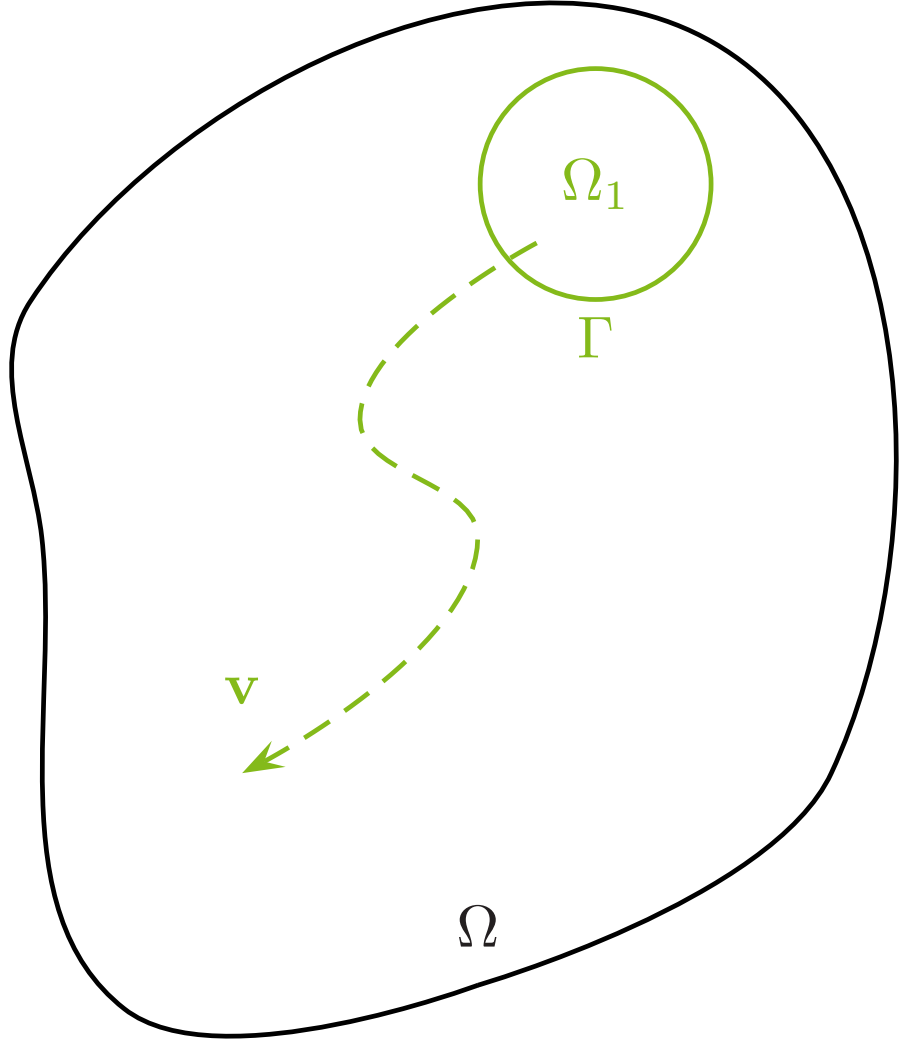
$$\frac{\partial \varphi}{\partial t} + \nabla \varphi \cdot \mathbf{v} = 0$$

- Initialization by a *signed distance function* (SDF):

$$\varphi(\mathbf{x}, 0) = \pm \text{dist}(x, \Gamma_0)$$

- + easy reconstruction of normal  $\mathbf{n} = \nabla \varphi$  and curvature  $\kappa = -\nabla \cdot \mathbf{n}$
- + ensures smoothness of  $\varphi$  and prevents instabilities ( $|\nabla \varphi| = 1$  a. e.)
- + can be used as interface proximity indicator

- SDF property is generally lost when solving numerically  $\rightarrow$  reinitialization required to "repair"  $\varphi$
- ! Reinitialization methods should not displace the interface



## PDE-based Post-Processing Methods

- Hyperbolic Reinitialization [HR]

- Solve artifical transport equation to steady state [5]

$$\frac{\partial \varphi}{\partial \tau} = S(\tilde{\varphi})(1 - |\nabla \varphi|), \quad \text{where} \quad S(\varphi) = \text{sign}(\varphi) \approx S_\varepsilon(\varphi) := \frac{\varphi}{\sqrt{\varphi^2 + \varepsilon^2}}$$

- Requires a stabilized numerical method

- Elliptic Reinitialization [ER]

- Minimize potential of Eikonal equation residual [1]

$$\min \mathcal{R}(\varphi) = \frac{1}{2} \int_{\Omega} p(|\nabla \varphi| - 1)^2 d\mathbf{x} + \frac{\alpha}{2} \int_{\Gamma(\tilde{\varphi})} \varphi^2 ds, \quad \text{here: } p(s) := s^2$$

- Linearized optimality conditions yield a Poisson-type system to be solved iteratively

$$\int_{\Omega} \left(1 - \frac{1}{|\nabla \varphi|}\right) \nabla \varphi \cdot \nabla v d\mathbf{x} + \alpha \int_{\Gamma(\tilde{\varphi})} \varphi v ds = 0$$

## Reinitialization Methods

- Hyperbolic Reinitialization
- Elliptic Reinitialization
- Parabolic Reinitialization
- Least Squares Reinitialization
- Optimal Control for Reinitialization
- Convected Reinitialization
- Brute-force Reinitialization
- Fast Marching
- Fast Sweeping

## PDE-based Optimal Control Approach

- Add bilinear source term to level set equation to correct  $\varphi$

$$\min J(\varphi, u) = \frac{1}{2} \| |\nabla \varphi| - 1 \|_{L^2(\Omega)}^2 + \frac{\beta}{2} \| u \|_{\mathcal{U}}^2 \quad \text{s. t.} \quad \frac{\partial \varphi}{\partial t} + \mathbf{v} \cdot \nabla \varphi + \varphi u = 0$$

- Linearized optimality conditions of first order are solved [2]

## Numerical results – rotation of a disk

- Disk subject to counter clockwise rotation

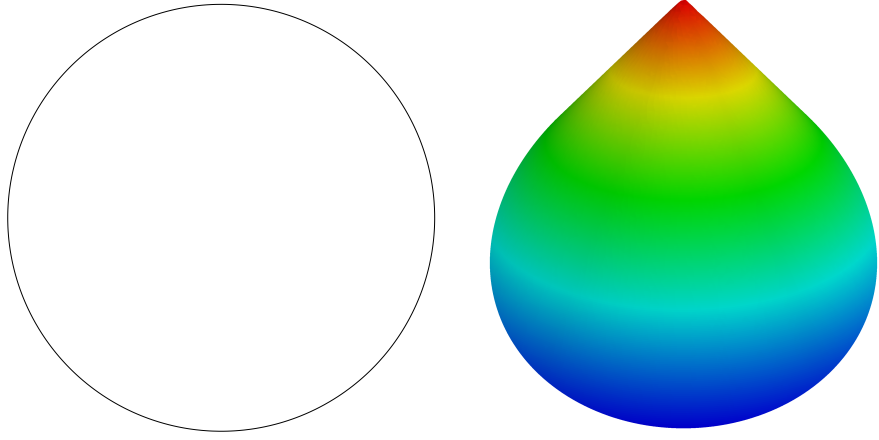
- Discretization: linear finite elements in space, Crank-Nicolson in time

refinement level $\ell$	0	1	2	3
$h_{\max}$	$\frac{\pi}{75}$	$\frac{\pi}{150}$	$\frac{\pi}{300}$	$\frac{\pi}{600}$
$\Delta t$	$\frac{\pi}{150}$	$\frac{\pi}{300}$	$\frac{\pi}{600}$	$\frac{\pi}{1200}$

- Evaluation after one full counter-clockwise revolution ( $t_{\text{final}} = 2\pi$ )

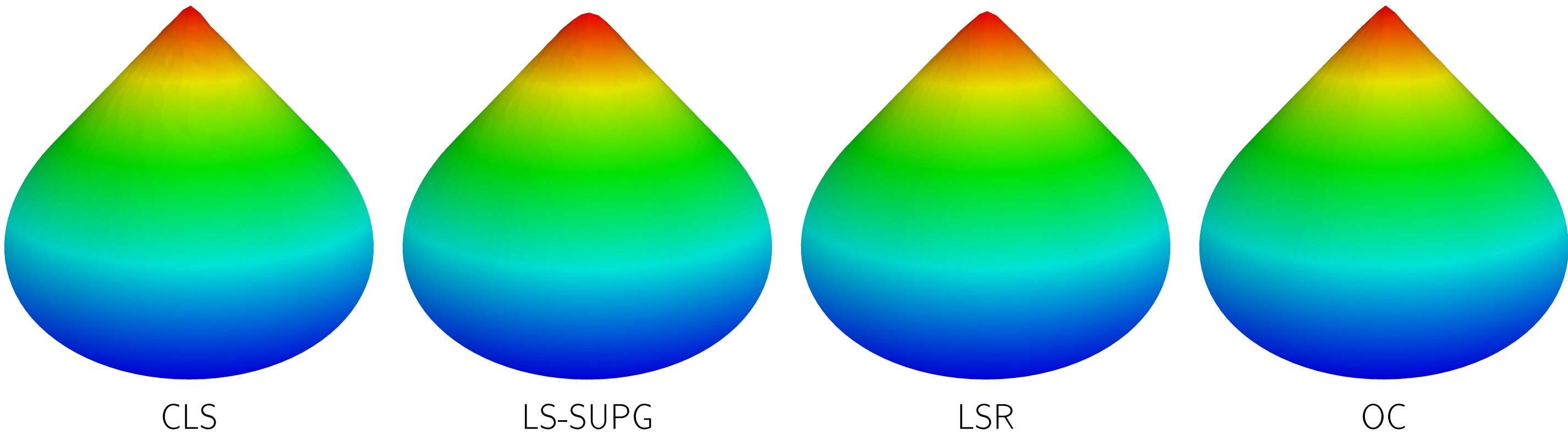
- Quantities of interest:  $L^2$  error, mean interface displacement  $e_1$  and residual of Eikonal equation  $e_E$  (in  $L^2(\Omega)$ )

$$e_1 = \frac{1}{N_{\Gamma}} \sum_{x_i \in P(\Gamma)} |D(\mathbf{x}_i) - \varphi_i|, \quad e_E = \| |\nabla \varphi| - 1 \|_{L^2(\Omega)}$$



error $\ell$	$L^2$			$e_1$			$e_E$		
	0	1	2	0	1	2	0	1	2
CLS	$7.89 \cdot 10^{-4}$	$9.73 \cdot 10^{-5}$	$2.89 \cdot 10^{-5}$	$1.21 \cdot 10^{-3}$	$1.38 \cdot 10^{-4}$	$3.13 \cdot 10^{-5}$	$1.01 \cdot 10^{-3}$	$2.84 \cdot 10^{-4}$	$7.88 \cdot 10^{-5}$
ER	$6.66 \cdot 10^{-4}$	$2.58 \cdot 10^{-4}$	$1.18 \cdot 10^{-4}$	$7.59 \cdot 10^{-4}$	$2.23 \cdot 10^{-4}$	$7.74 \cdot 10^{-5}$	$5.57 \cdot 10^{-4}$	$1.16 \cdot 10^{-4}$	$3.01 \cdot 10^{-5}$
LS	$1.35 \cdot 10^{-3}$	$5.10 \cdot 10^{-4}$	$1.87 \cdot 10^{-4}$	$2.26 \cdot 10^{-3}$	$6.67 \cdot 10^{-4}$	$2.63 \cdot 10^{-4}$	$1.81 \cdot 10^{-2}$	$9.54 \cdot 10^{-3}$	$5.31 \cdot 10^{-3}$
LS-SUPG	$6.02 \cdot 10^{-4}$	$2.32 \cdot 10^{-4}$	$7.98 \cdot 10^{-5}$	$4.10 \cdot 10^{-4}$	$9.61 \cdot 10^{-5}$	$2.65 \cdot 10^{-5}$	$1.91 \cdot 10^{-3}$	$6.58 \cdot 10^{-4}$	$2.19 \cdot 10^{-4}$
LSR	$6.83 \cdot 10^{-4}$	$2.82 \cdot 10^{-4}$	$1.54 \cdot 10^{-4}$	$5.70 \cdot 10^{-4}$	$3.40 \cdot 10^{-4}$	$1.06 \cdot 10^{-4}$	$9.54 \cdot 10^{-4}$	$1.89 \cdot 10^{-4}$	$3.01 \cdot 10^{-5}$
OC	$3.95 \cdot 10^{-4}$	$1.66 \cdot 10^{-4}$	$4.90 \cdot 10^{-4}$	$5.68 \cdot 10^{-4}$	$1.35 \cdot 10^{-4}$	$3.89 \cdot 10^{-5}$	$5.13 \cdot 10^{-4}$	$1.23 \cdot 10^{-4}$	$3.17 \cdot 10^{-5}$
PR	$6.61 \cdot 10^{-4}$	$2.99 \cdot 10^{-4}$	$9.58 \cdot 10^{-5}$	$5.36 \cdot 10^{-4}$	$2.65 \cdot 10^{-4}$	$7.75 \cdot 10^{-5}$	$5.66 \cdot 10^{-4}$	$1.49 \cdot 10^{-4}$	$4.19 \cdot 10^{-5}$

- Plots at refinement level  $\ell = 0$  ( $\sim 1200$  dofs) and  $t = t_{\text{final}}$



## PDE-based All-At-Once Methods

- Convected Level Set Method [CLS]

- Embed reinitialization into transport equation [6]

$$\frac{\partial \varphi}{\partial t} + (\mathbf{v} + \lambda \mathbf{w}) + \lambda S(|\nabla \varphi| - 1) = 0$$

- Requires a stabilized numerical method

- Least Squares reinitialization [LSR]

- Solve a least-squares problem with incorporated Eikonal residual as penalty term

$$\min J(\varphi) = \frac{1}{2} \left\| \frac{\partial \varphi}{\partial t} + \mathbf{v} \cdot \nabla \varphi \right\|_{L^2(\Omega)}^2 + \frac{\alpha}{2} \| |\nabla \varphi| - 1 \|_{L^2(\Omega)}^2$$

- Built-in stabilization

- Parabolic Reinitialization [PR]

- Based on elliptic reinitialization,  $\mathcal{R}(\varphi)$  is directly added to the transport equation:

$$\int_{\Omega} \left( \frac{\partial \varphi}{\partial t} + \nabla \varphi \cdot \mathbf{v} \right) d\mathbf{x} + \delta \mathcal{R}(\varphi, v) + \beta \sum_K \int_{\partial K} [\mathbf{n} \cdot \nabla \varphi] [\mathbf{n} \cdot \nabla v] ds$$

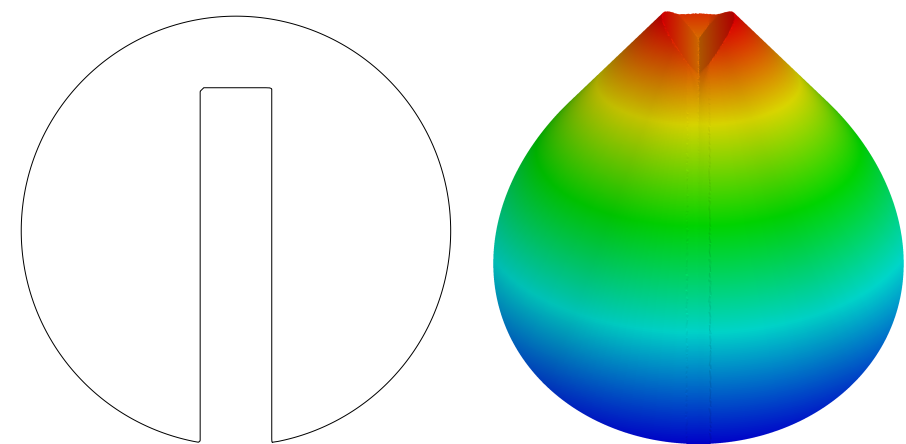
- Additional CIP regularization term [3] added

## Numerical results – rotation of a slotted disk

- Zalesak's slotted disk subject to counter clockwise rotation

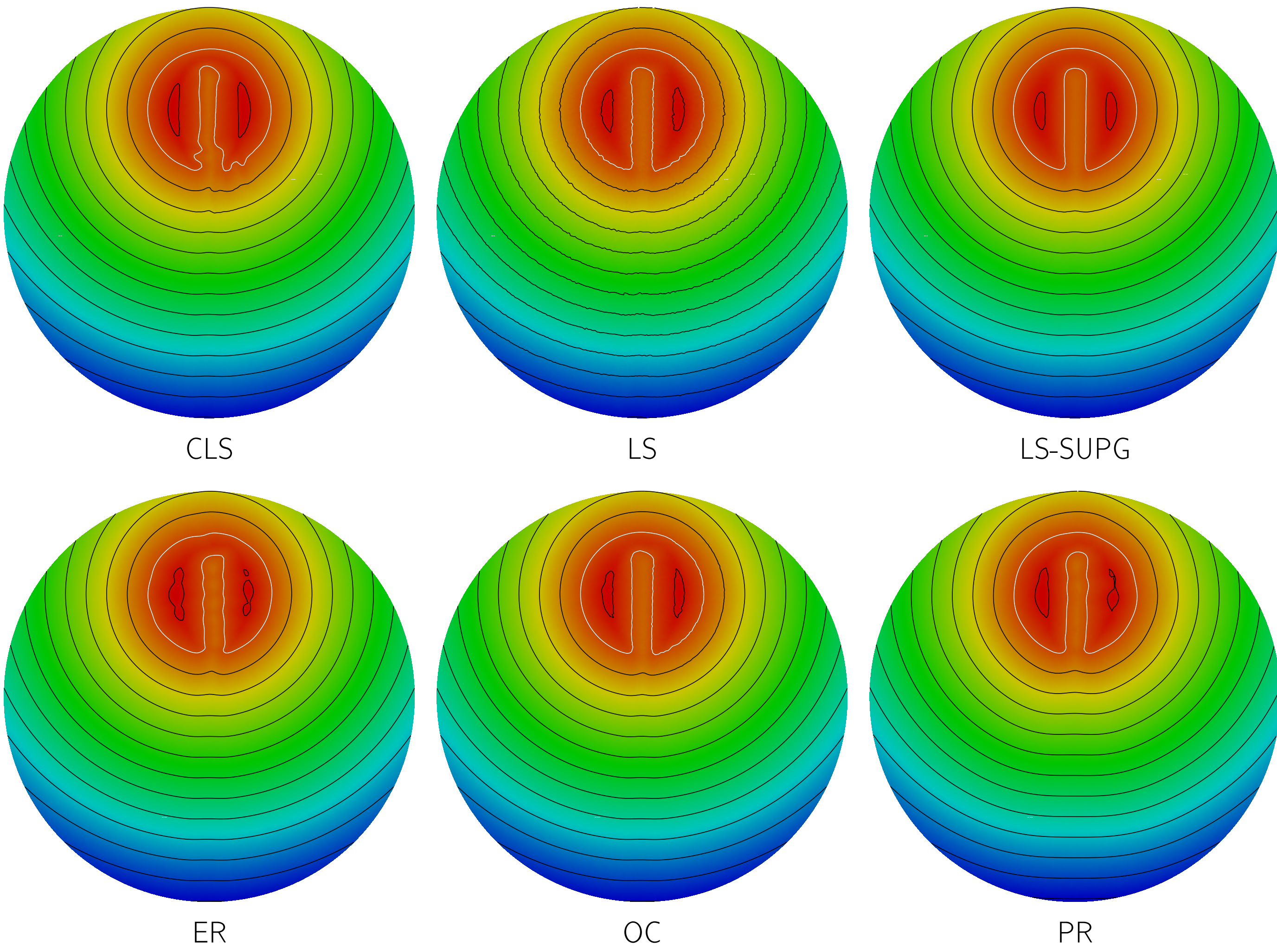
- Discretization: linear finite elements in space, Crank-Nicolson in time

- Evaluation after one full counter-clockwise revolution



error $\ell$	$L^2$			$e_1$			$e_E$		
	1	2	3	1	2	3	1	2	3
CLS	$2.88 \cdot 10^{-3}$	$1.07 \cdot 10^{-3}$	$2.29 \cdot 10^{-4}$	$6.10 \cdot 10^{-3}$	$2.56 \cdot 10^{-3}$	$4.60 \cdot 10^{-4}$	$1.00 \cdot 10^{-2}$	$4.85 \cdot 10^{-5}$	$2.52 \cdot 10^{-3}$
LS	$1.47 \cdot 10^{-3}$	$7.08 \cdot 10^{-4}$	$3.36 \cdot 10^{-4}$	$2.92 \cdot 10^{-3}$	$1.21 \cdot 10^{-3}$	$5.36 \cdot 10^{-4}$	$6.18 \cdot 10^{-2}$	$4.62 \cdot 10^{-2}$	$3.93 \cdot 10^{-2}$
LS-SUPG	$9.52 \cdot 10^{-4}$	$4.65 \cdot 10^{-4}$	$2.21 \cdot 10^{-4}$	$1.84 \cdot 10^{-3}$	$7.18 \cdot 10^{-4}$	$1.51 \cdot 10^{-4}$	$1.12 \cdot 10^{-2}$	$7.05 \cdot 10^{-3}$	$5.17 \cdot 10^{-3}$
LSR	$3.94 \cdot 10^{-3}$	$2.08 \cdot 10^{-3}$	$9.51 \cdot 10^{-4}$	$9.25 \cdot 10^{-3}$	$5.26 \cdot 10^{-3}$	$2.21 \cdot 10^{-3}$	$4.05 \cdot 10^{-3}$	$1.88 \cdot 10^{-3}$	$9.02 \cdot 10^{-4}$
OC	$1.05 \cdot 10^{-3}$	$5.22 \cdot 10^{-4}$	$1.74 \cdot 10^{-4}$	$2.70 \cdot 10^{-3}$	$7.86 \cdot 10^{-4}$	$2.78 \cdot 10^{-4}$	$1.06 \cdot 10^{-2}$	$6.41 \cdot 10^{-3}$	$3.42 \cdot 10^{-3}$
PR	$3.68 \cdot 10^{-3}$	$2.15 \cdot 10^{-3}$	$1.62 \cdot 10^{-3}$	$4.27 \cdot 10^{-3}$	$2.27 \cdot 10^{-3}$	$1.53 \cdot 10^{-3}$	$3.76 \cdot 10^{-3}$	$1.72 \cdot 10^{-3}$	$9.47 \cdot 10^{-3}$

- Isocontours at refinement level  $\ell = 2$  ( $\sim 18000$  dofs) and  $t = t_{\text{final}}$



## References

- [1] Christopher Basting and Dmitri Kuzmin. A minimization-based finite element formulation for interface-preserving level set reinitialization. *Computing*, 95(1):13–25, 2013. ISSN 0010-485X. doi: 10.1007/s00607-012-0259-z. URL <http://dx.doi.org/10.1007/s00607-012-0259-z>.
- [2] Christopher Basting and Dmitri Kuzmin. Optimal control for mass conservative level set methods. *Journal of Computational and Applied Mathematics*, 270:343 – 352, 2014. ISSN 0377-0427. doi: <http://dx.doi.org/10.1016/j.cam.2013.12.040>. URL <http://www.sciencedirect.com/science/article/pii/S0377042713007139>.
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- [4] Daniel Hartmann, Matthias Meinke, and Wolfgang Schröder. Differential equation based constrained reinitialization for level set methods. *Journal of Computational Physics*, 227(14):6821 – 6845, 2008. ISSN 0021-9991. doi: <http://dx.doi.org/10.1016/j.jcp.2008.03.040>. URL <http://www.sciencedirect.com/science/article/pii/S0021999108001964>.
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- [6] Laurence Ville, Luisa Silva, and Thierry Coupez. Convected level set method for the numerical simulation of fluid buckling. *International Journal for Numerical Methods in Fluids*, 66(3):324–344, 2011. ISSN 1097-0363. doi: 10.1002/ld.2259. URL <http://dx.doi.org/10.1002/ld.2259>.