

An interface fitted Arbitrary Lagrangian-Eulerian finite element approach for free and moving boundary problems

Setting and motivation

Model problem with moving (sharp) interface:

$$\begin{cases} \partial_t u + A[\Omega, \Gamma; u] = f & \text{on } \Omega(t) = \Omega_1(t) \cup \Omega_2(t) \cup \Gamma(t) \\ \text{some BCs} & \text{on } \partial\Omega \end{cases}$$

■ Two classes of numerical methods for moving boundary problems:

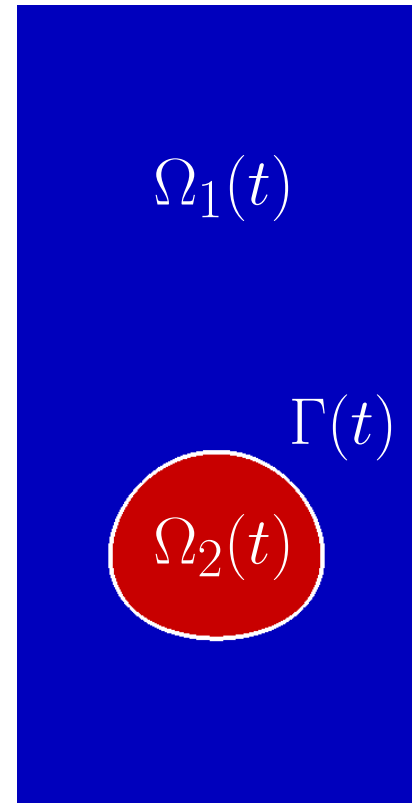
- **fixed mesh** (e.g. level set / VOF based methods \Rightarrow interface capturing methods)
- **deforming mesh** (e.g. ALE based methods following the evolution of $\Gamma(t)$ with underlying mesh \Rightarrow interface tracking methods)

■ We would like to use a **deforming mesh** method:

- + simple definition / implementation of “problem-tailored” FE spaces
- + high accuracy / nice approximation properties

– Crucial point: strong deformation of $\Gamma \Rightarrow$ **how to handle mesh deformation?**

■ Typical for **Arbitrary Lagrangian-Eulerian (ALE)** methods: restate the problem on a *fixed* reference domain $\hat{\Omega}$ using an **ALE mapping** $\chi(t) : \hat{\Omega} \rightarrow \Omega(t)$ and **ALE velocity** $\hat{\mathbf{w}}(t, \hat{\mathbf{x}}) = \partial_t \chi(t, \hat{\mathbf{x}})$



$$\begin{cases} \partial_t \hat{u} - \hat{\mathbf{w}} \cdot \hat{\nabla} \hat{u} + \hat{A}[\hat{\Omega}, \hat{\Gamma}; \hat{u}] = \hat{f} & \text{on } \hat{\Omega} \\ \text{some BCs} & \text{on } \partial\hat{\Omega} \end{cases}$$

■ **Problem** (standard ALE methods): discrete ALE mappings are **continuous in time** (usually obtained from an extension of the interface displacement) \Rightarrow construction of mappings becomes difficult as deformation increases

■ **Key idea**: use **discontinuous in time** ALE mappings: “re-parametrize” the ALE mapping and try to obtain **interface aligned meshes of optimal quality, without changing connectivity**

■ Similar approaches: Gawlik, Frei/Richter, Codina, ...

Interface aligned meshes of optimal quality

Given a triangulation $\hat{\mathcal{T}}$ of the reference domain $\hat{\Omega}$, find a deformed triangulation $\mathcal{T}^* = \chi^*(\hat{\mathcal{T}})$ such that

■ The ALE mapping χ^* is an **admissible deformation**:

$$\chi^* \in \mathcal{D} := \left\{ \eta : \hat{\Omega} \rightarrow \Omega \mid \eta \in \left(C^0(\hat{\Omega}) \right)^n, \nabla \eta|_{\hat{T}} \in \text{GL}(n), \det(\nabla \eta|_{\hat{T}}) > 0 \quad \forall \hat{T} \in \hat{\mathcal{T}} \right\}$$

■ \mathcal{T}^* is “**optimal**”^[7] in the sense that χ^* minimizes a functional $\mathcal{F} : \mathcal{D} \rightarrow \mathbb{R}$:

- \mathcal{F} is expressed as sum of element-wise contributions $\mathcal{F}(\chi) = \sum_{\hat{T} \in \hat{\mathcal{T}}} \mu_{\hat{T}} F_{\hat{T}}(\chi)$, where $F_{\hat{T}} : \chi|_{\hat{T}} \rightarrow \mathbb{R}$
- Under the assumptions of isotropy, translation and frame indifference, $F_{\hat{T}}$ takes on the form

$$F_{\hat{T}}(\chi) = F \left(\|\nabla R_{\hat{T}}(\chi)\|_F^2, \|\text{Cof}(\nabla R_{\hat{T}}(\chi))\|_F^2, \det(\nabla R_{\hat{T}}(\chi)) \right),$$

where $R_{\hat{T}}(\chi)$ denotes the affine mapping from a *desired* reference element to $\chi|_{\hat{T}}$

◦ **Key property: minimizers of \mathcal{F} yield non-degenerate triangulations**

■ \mathcal{T}^* is **aligned** with Γ :

◦ Introduce an auxiliary *level set function* $\phi : [0, T] \times \Omega \rightarrow \mathbb{R}$:

$$\Omega_{1/2}(t) = \{\mathbf{x} \in \Omega : \phi(t, \mathbf{x}) \geq 0\}, \quad \Gamma(t) = \{\mathbf{x} \in \Omega : \phi(t, \mathbf{x}) = 0\}.$$

◦ Observation: an edge e of \mathcal{T} is intersected by Γ if and only if

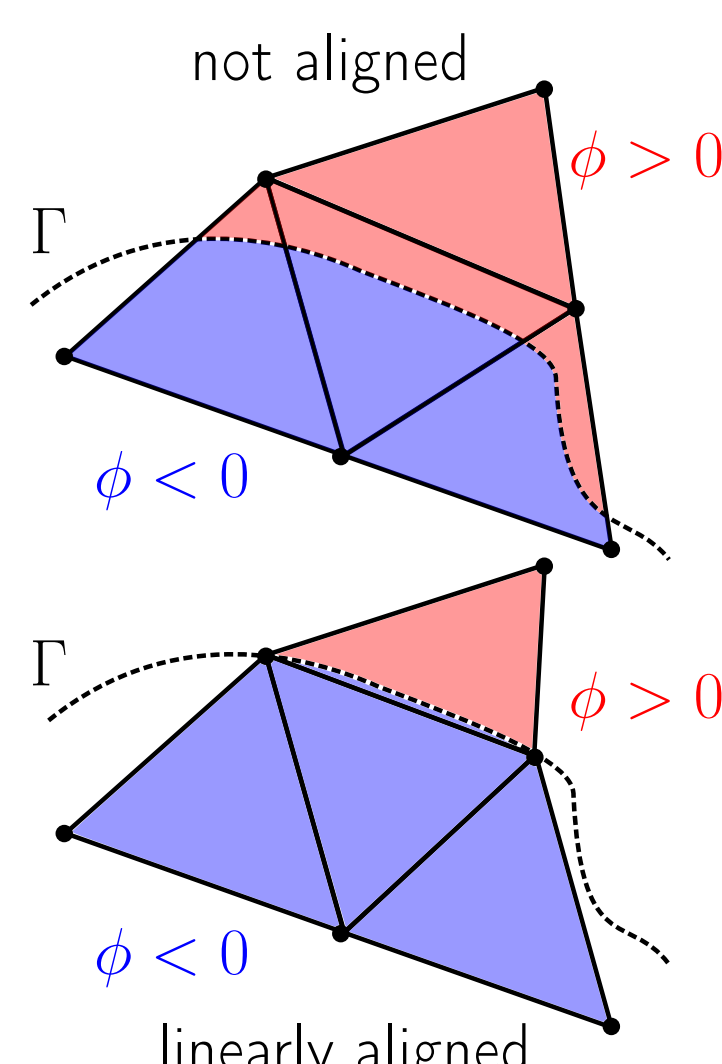
$$\phi(\mathbf{x}_{e,1})\phi(\mathbf{x}_{e,2}) < 0 \quad (1)$$

where $\mathbf{x}_{e,1}$ and $\mathbf{x}_{e,2}$ are the endpoints of $e \Rightarrow$ restrict \mathcal{D} to deformations for which no e fulfills (1)

◦ A deformed triangulation is **linearly aligned** if and only if

$$0 = c(\chi) = \sum_{e \in \chi(\hat{\mathcal{T}})} \mathcal{H}(\phi(\mathbf{x}_{e,1})\phi(\mathbf{x}_{e,2})) \quad \text{where} \quad \mathcal{H}(z) : \begin{cases} > 0 & \text{for } z < 0 \\ = 0 & \text{for } z \geq 0. \end{cases}$$

◦ Higher order approximation (isoparametric elements/blending etc.) achievable through “simple post-processing”



Optimal, (linearly) aligned ALE mappings are obtained from the nonlinear constrained optimization problem

$$\underbrace{\min \mathcal{F}(\chi)}_{\text{optimize mesh quality}} \quad \text{s.t.} \quad \underbrace{c(\chi) = 0}_{\text{resulting mesh is aligned with } \Gamma} \quad (2)$$

Outline of the method

Assume that at time instant t_j , approximations of the interface Γ_h^j , the domain Ω_h^j and a numerical solution u_h^j are known. Proceed as follows:

■ **Step 1**: Check if re-parametrization of current geometry is necessary

- **No** \Rightarrow Set $\tilde{u}_h^j := u_h^j$ and let $\tilde{\Gamma}_h^j := \Gamma_h^j$, $\tilde{\Omega}_h^j := \Omega_h^j$
- **Yes** \Rightarrow * Obtain new parametrized domain $\tilde{\Omega}_h^j$ (as well as $\tilde{\Gamma}_h^j$) from mesh optimization problem (2).
* Project/interpolate solution onto new domain, $\tilde{u}_h^j := I_{\Omega_h^j \rightarrow \tilde{\Omega}_h^j}(u_h^j)$

■ **Step 2**: Compute the solution u_h^{j+1} at time t_{j+1} using \tilde{u}_h^j as initial value

■ **Step 3**: Obtain new boundary position Γ_h^{j+1}

■ **Step 4**: Get Ω_h^{j+1} from new boundary position Γ_h^{j+1} and (standard) extension operator, i.e. $\Omega_h^{j+1} := E(\Gamma_h^{j+1})$

Steps 2 to 4 are “standard ALE” steps \Rightarrow the strategy reduces to a standard ALE approach when **Step 1** is omitted

Two-phase flows with surface tension

The model

- Ω_1 and Ω_2 are occupied by two immiscible Newtonian fluids (different densities / viscosities), modeled by the incompressible Navier-Stokes equations
- Boundary conditions on Γ :

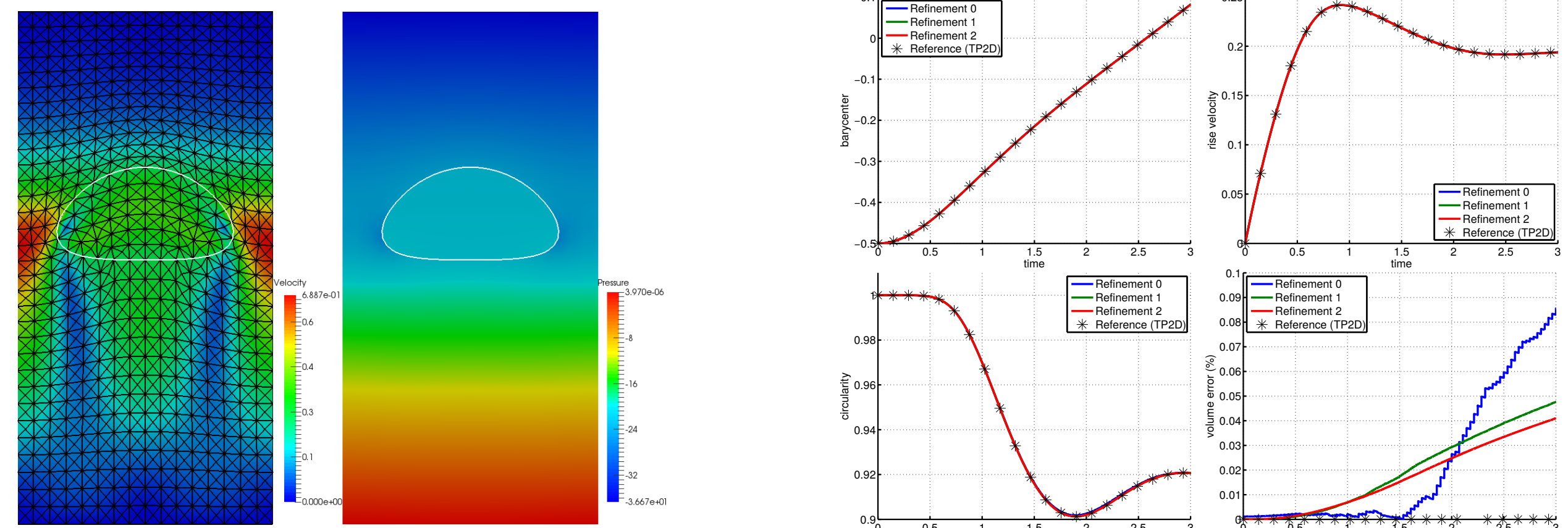
$$\begin{aligned} \text{continuity of velocities} & \quad [\mathbf{u}] = 0 \\ \text{balance of forces} & \quad [\sigma \mathbf{n}] = \frac{1}{\text{We}} \kappa \mathbf{n} \\ \text{interface evolution law} & \quad \mathbf{u}_\Gamma = \mathbf{u} \cdot \mathbf{n} \end{aligned}$$

Numerical realization^[2]

- Quadratic Taylor-Hood finite elements
- Isoparametric approximation Γ_h of Γ
- Discontinuous pressure approximation across Γ_h . Numerical realization using a discrete projection (SPM^[3]).
- Semi-implicit treatment of surface tension \Rightarrow unconditional stability:

$$\frac{1}{\text{We}} \int_\Gamma \kappa \mathbf{n} \cdot \boldsymbol{\varphi} \approx \frac{1}{\text{We}} \int_{\Gamma_h} \nabla_h \cdot (\mathbf{x}^j + \tau \mathbf{u}_h^{j+1}) : \nabla_h \boldsymbol{\varphi}_h$$

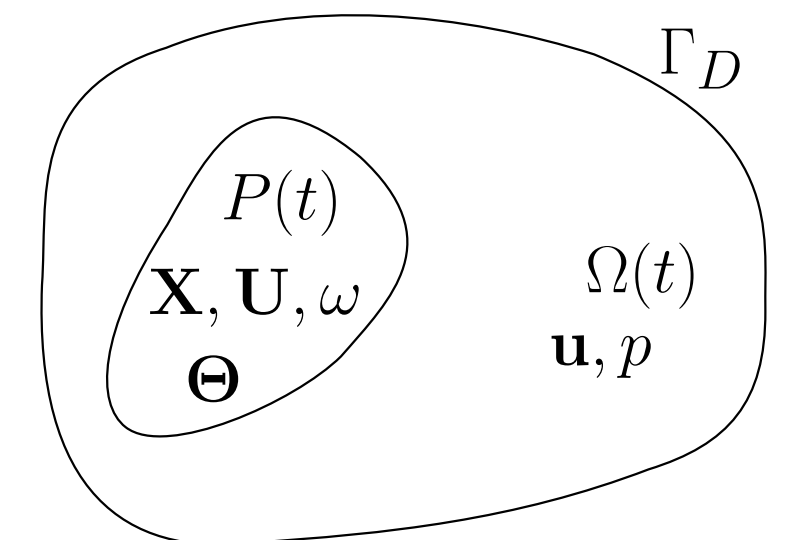
Rising bubble benchmark^[5]



Particulate flows

The model

- $\Omega(t)$ occupied by a Newtonian fluid, modeled by the incompressible Navier-Stokes equations
- Rigid particle $P(t)$ described by position \mathbf{X} , orientation in space $\boldsymbol{\Theta}$, translational velocity \mathbf{U} and angular velocity ω
- Combined fluid/particle domain: $\Omega_c = P(t) \cup \partial P(t) \cup \Omega(t)$



Outline of solution strategy^[1]

■ Based on a *fictitious domain approach*^[4]: formulation of the problem in the space of “combined” velocities H_c ,

$$H_c(\Omega_c) = \left\{ \mathbf{v} = (\mathbf{v}, \mathbf{V}, \xi) \mid \mathbf{v} \in \left(H^1(\Omega_c) \right)^2, \mathbf{V} \in \mathbb{R}^2, \xi \in \mathbb{R}, \mathbf{v} = 0 \text{ on } \Gamma_D, \mathbf{v} = \mathbf{V} + \xi \times \mathbf{r} \text{ in } P(t) \right\}$$

■ Decoupling of the complex problem into a series of conceptually simple sub-problems per time step:

1. Velocity Verlet type predictor for new particle positions and velocities
2. **Adapt mesh to new particle positions**
3. BDF2 based projection scheme for the Navier-Stokes equations, SPM^[6] used to enforce rigid body motion

What to expect from particle aligned meshes?

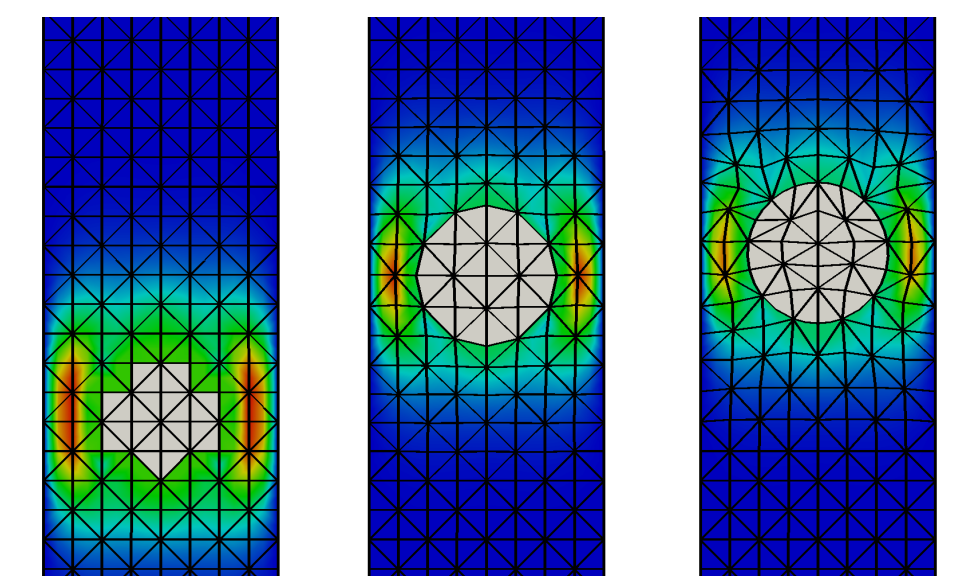
- Error analysis hinges on how well $P(t)$ and $H_c(\Omega_c)$ are resolved by the mesh and corresponding FE space
- A priori error estimate for (quasi-) stationary model problem:

$$\|\mathbf{u} - \mathbf{u}_h\|_c \leq C(\mathbf{u}) \begin{cases} h^{1/2} & \text{no mesh alignment} \\ h & \text{linear mesh alignment} \\ h^2 & \text{quadratic mesh alignment} \end{cases}$$

	No alignment		Linear alignment		Quadratic alignment	
Level	$\ \mathbf{u} - \mathbf{u}_h\ _c$	EOC	$\ \mathbf{u} - \mathbf{u}_h\ _c$	EOC	$\ \mathbf{u} - \mathbf{u}_h\ _c$	EOC
4	0.35533e+00	0.58	0.14603e+00	0.99	0.10622e-01	1.91
5	0.27877e+00	0.35	0.73734e-01	0.99	0.27702e-02	1.94
6	0.19231e+00	0.54	0.36833e-01	1.00	0.71143e-03	1.96

■ Sedimentation of a particle: error in terminal velocity

	No alignment		Linear alignment		Quadratic alignment	
Level	U	Err (%)	U	Err (%)	U	Err (%)
3	6.5725e-01	54.19	4.5138e-01	5.89	4.1848e-01	1.83
4	4.7072e-01	10.43	4.3292e-01	1.56	4.2402e-01	0.53
5	4.4254e-01	3.82	4.2772e-01	0.34	4.2552e-01	0.18
6	4.3917e-01	3.03	4.2609e-01	0.099	4.2614e-01	0.032
7	4.3487e-01	2.02	4.2641e-01	0.033	4.2627e-01	—



Fluid-structure interaction

The model

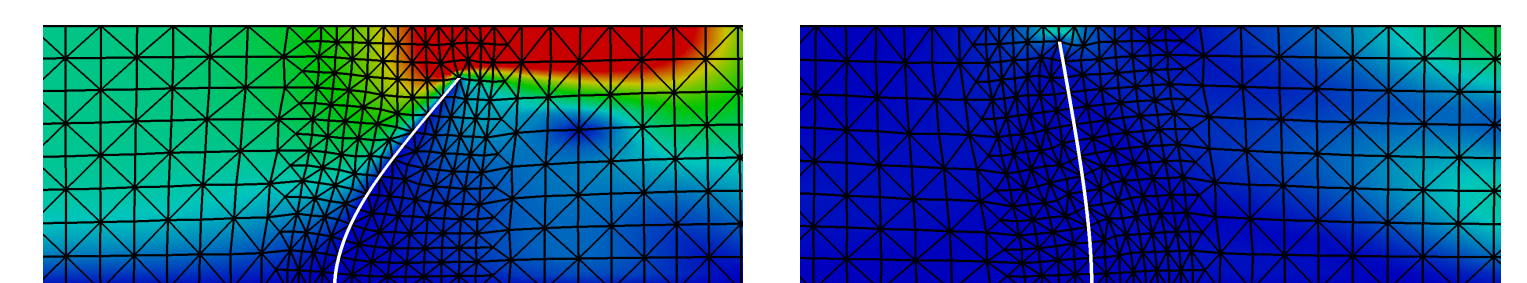
- Simulate a **thin, elastic structure** Γ immersed in an incompressible, viscous, Newtonian fluid.
- Γ is modeled by a (nonlinear) beam equation. Coupling condition reads:

$$\begin{aligned} \rho_s \ddot{\mathbf{x}} + EI \mathbf{x}''' &= [\sigma \mathbf{n}] \quad \text{on } \Gamma \\ (|\mathbf{x}'| &= 1 \quad \text{inextensible model}) \end{aligned} \quad (3)$$

■ We consider *closed* (balloon-type) and *open* structures

Challenges and numerical realization

- Strongly coupled, partitioned approach: two separate solvers for structure and fluid dynamics
- **Important**: *Added mass effect* becomes dominant for small structure densities $\rho_s \Rightarrow$ use a **Robin-Neumann coupling** (partially embedding (3) into fluid problem)



References

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- [6] Rodolphe Prignitz and Eberhard Bänsch. Particulate flows with the subspace projection method. *Journal of Computational Physics*, 260:249–272, 2014.
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