Anisotropic slope limiting in discontinuous Galerkin methods for transport equations

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Slope limiting in 2D

- Slope limiters
  - enforce geometric maximum principles at certain control points
  - constrain derivatives in the Taylor expansion
  - should be able to recognize possible anisotropies and smooth extrema
- Vertex based limiter
  - imposes less restrictive constraints than face-based limiters
  - limiting at the center of mass limits derivatives in a hierarchical manner
  - all derivatives of the same degree are multiplied by the same correction factor $\Rightarrow$ inappropriate in the presence of strong directional derivatives
- Our approach
  - anisotropic version of the vertex based limiter
  - simple closed-form expressions

Model problem

- Linear convection equation

$$\frac{\partial u}{\partial t} + \nabla \cdot (\mathbf{v}u) = 0 \qquad \text{in } \Omega.$$

- Initial condition

$$u(\cdot, 0) = u_0 \qquad \text{in } \Omega.$$

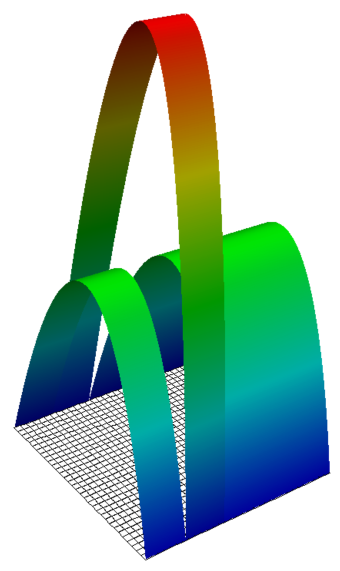
- Boundary condition

$$(\mathbf{v}u)\cdot\mathbf{n} = (\mathbf{v}u_{\text{in}})\cdot\mathbf{n} \qquad \text{on } \Gamma_{\text{in}} = \{\mathbf{x}\in\Gamma \,|\, \mathbf{v}(\mathbf{x})\cdot\mathbf{n}(\mathbf{x}) < 0\}.$$

- Convection in a unit square $\Omega = (0,1)\times(0,1)$, $\mathbf{v}(x,y) = (0,1)$

$$u_0(x,y) = w(x)4y(1-y),$$

$$w(x) = \begin{cases} 2, & \text{if } 0.2 \le x \le 0.4, \\ 1, & \text{otherwise.} \end{cases}$$



Discontinuous Galerkin method

- Local variational formulation

$$\int_K \left(w\frac{\partial u}{\partial t} - \nabla w \cdot \mathbf{v}u \right) \mathrm{d}\mathbf{x} + \int_{\partial K} w\hat{u}\mathbf{v}\cdot\mathbf{n}\,\mathrm{d}s = 0, \qquad \forall w \in V,$$

$$\hat{u}(\mathbf{x},t) = \begin{cases} \lim\limits_{\epsilon\to+0} u(\mathbf{x}+\epsilon\mathbf{n},t) & \text{if } \mathbf{x}\in\partial K_{\text{in}}\setminus\Gamma_{\text{in}}, \\ u_{\text{in}}(\mathbf{x},t) & \text{if } \mathbf{x}\in\partial K_{\text{in}}\cap\Gamma_{\text{in}}, \\ \lim\limits_{\epsilon\to-0} u(\mathbf{x}+\epsilon\mathbf{n},t) & \text{otherwise.} \end{cases}$$

- Finite element approximation

$$u_h(\mathbf{x},t) = \sum_{j=1}^{N} u_j(t)\varphi_j(\mathbf{x}).$$

- Time discretization, e.g., using an explicit Runge-Kutta method.

Isotropic vertex based limiter

- Piecewise-linear DG approximation, linear shape function

$$u_h(x,y) = u_0 + u_x(x-x_0) + u_y(y-y_0).$$

- Inequality constraints
  - linear components of u_h can increase the jumps of the DG solution
  - may need to be limited

$$\bar{u}_h(x,y) = u_0 + \alpha_x u_x(x-x_0) + \alpha_y u_y(y-y_0).$$

  - inequality constraints

$$u_i^{\min} \le \bar{u}_h(x_i,y_i) \le u_i^{\max}, \qquad i = 1,\dots,M.$$

- Isotropic limiting
  - same correction factor $\alpha_x = \alpha_y = \alpha$, Barth-Jesperson formula

$$\alpha = \min_{1\le i\le M} \begin{cases} \min\left\{1, \frac{u_i^{\max}-u_0}{u_i-u_0}\right\}, & \text{if } u_i - u_0 > 0, \\ 1, & \text{if } u_i - u_0 = 0, \\ \min\left\{1, \frac{u_i^{\min}-u_0}{u_i-u_0}\right\}, & \text{if } u_i - u_0 < 0. \end{cases}$$

  - may give rise to unnecessary cancellation of smooth directional derivatives

Anisotropic vertex based limiter

- Directional prelimiting

$$u_i^{\min} - u_0 \le \beta_{x,i}u_x(x_i-x_0) \le u_i^{\max} - u_0,$$
$$u_i^{\min} - u_0 \le \beta_{y,i}u_y(y_i-y_0) \le u_i^{\max} - u_0,$$

to ensure that an increment in either x- or y-direction cannot create an undershoot or overshoot.

- Directional increments of opposite signs: $\alpha_{x,i} := \beta_{x,i}$, $\alpha_{y,i} := \beta_{y,i}$.
- Directional increments of the same sign:
  - further limiting required to keep the sum of increments within the bounds
  - optimal correction factors determined by minimizing the Euclidean norm of the difference between the unlimited and limited gradients

Arbitrary frame of reference

- Anisotropies not aligned with the axes of the Cartesian coordinate system.
- Direction aligned with the unconstrained gradient $\nabla u_h|_K$, $\boldsymbol{\xi} = \frac{\nabla u_h}{\|\nabla u_h\|} = (\xi_x, \xi_y)^T$ $\quad \boldsymbol{\eta} = (-\xi_y, \xi_x)^T$.
- Use of frame-invariant directions preferred, especially in extensions of the anisotropic limiter to vector fields.
- Application of the anisotropic slope limiter to the components of $(u_\xi, u_\eta)^T$ replacing $(u_x, u_y)^T$ by $\Phi(\theta)(u_x, u_y)^T$,

$$\Phi(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \alpha_\xi & 0 \\ 0 & \alpha_\eta \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

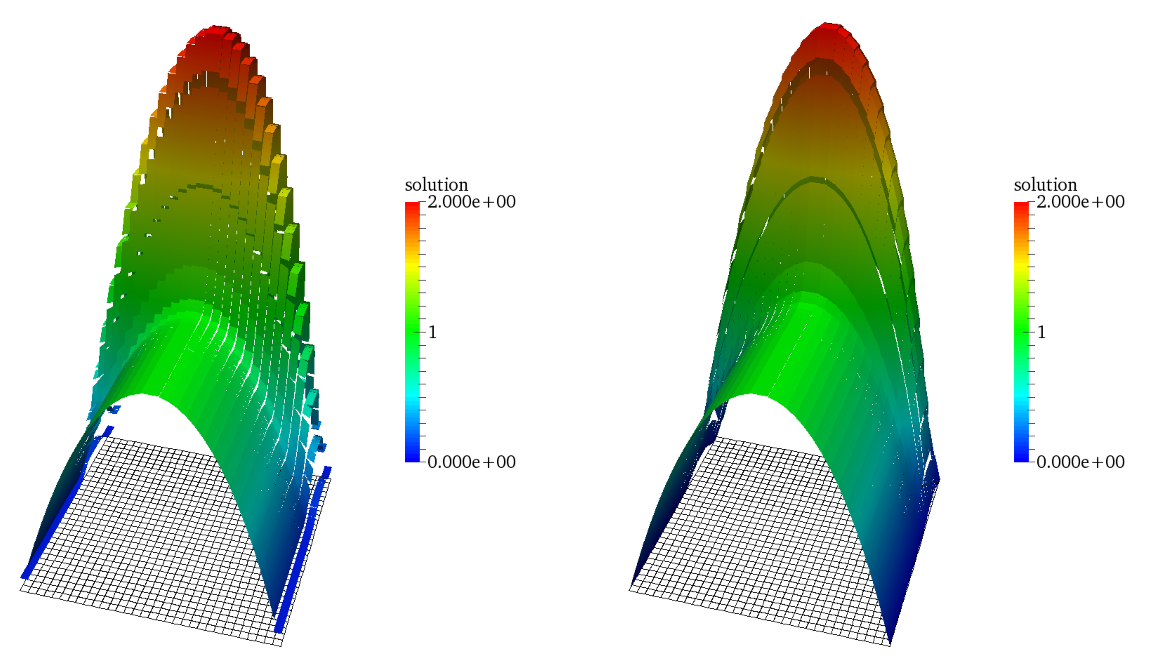
Correction factors α_ξ, α_η obtained as before but with a rotation of the local coordinate system.

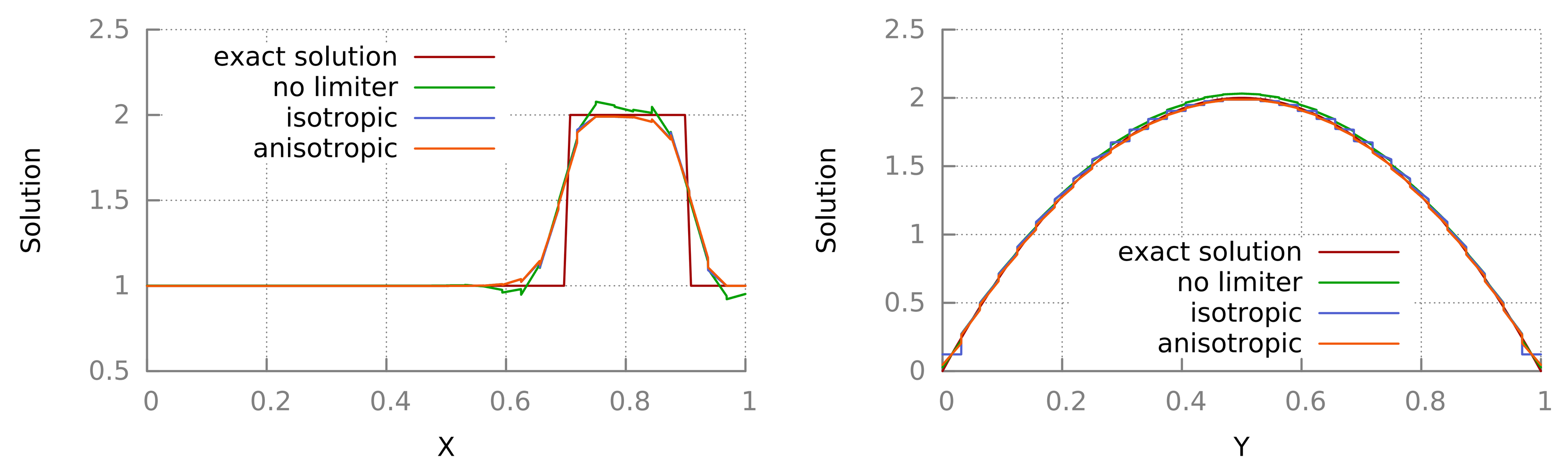
Treatment of boundary points

- Knowledge of the cell mean values is insufficient to construct usable bounds $u_i^{\max}$ and $u_i^{\min}$ for boundary points.
- Unnecessary cancellation of normal derivatives in boundary elements avoided by using the Dirichlet BC and/or extrapolated boundary values.
- We estimate the bounds for the boundary vertices lying outside the Dirichlet part of the boundary using the mean values not only from elements containing the vertex, but also from all adjacent boundary edges.
- We consider the boundary normal and tangential directions as the frame-invariant directions.

Numerical experiments

- Convection in a unit square, solution at time 0.5, isotropic (left) and anisotropic (right) vertex based limiter.





- Convection in a rotated square, Ω rotated by an angle $\pi/6$, $\mathbf{v}(x,y) = (\cos\frac{\pi}{6}, \sin\frac{\pi}{6})$