A fully-discrete high-order ALE method based on untwisted time-space control volumes

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Outline

- Motivation
- A fully-discrete high-order ALE method based on untwisted time-space control volume
  - Fully-discrete ALE method
  - Vorticity-free mesh generation
  - High order flux computation with GRP
- Numerical examples
- Concluding remarks

Motivation

Compressible flows

- Coming from a variety of scientific and engineering problems,
- Featured with strong shocks, contact discontinuities, free surfaces, interface instabilities and mixing processes…

ALE method plays a dominant role in the field of engineering research.

Some research focus of ALE method

- Mesh moving strategy
  - Important to computational resolution
  - Previous work: mesh moving strategy based on **spatial control volume**
  - This work: considering mesh moving strategy based on **time-space control volume**
- High order ALE both in time and space
  - multistage high order in time **Runge-Kutta**

    large computational expense
  - single-stage high order in time **GRP, ADER**
    - GRP(acoustic approximation) + Lagrangian scheme(Maire 2009)
    - ADER + one step ALE (Dumbser 2011)
    - GRP + one step ALE (this work)

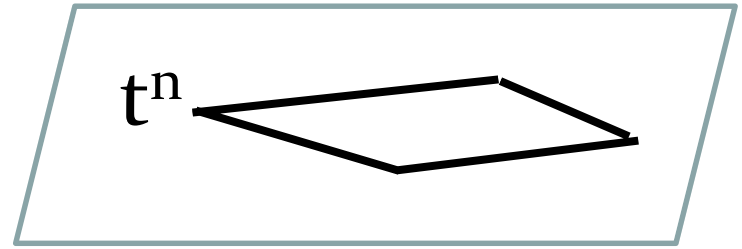
A fully-discrete high-order ALE method

- Discrete framework
  - Fully-discrete method
- ALE mesh
  - Vorticity-free mesh
- Flux computation
  - GRP solver

Semi-discrete ALE method

2-D Euler equations

$$W_t + F_x + G_y = 0,$$
$$W = (\rho, \rho u, \rho v, \rho E)^T,$$
$$F = (\rho u, \rho u^2 + p, \rho u v, \rho E u + p u)^T,$$
$$G = (\rho v, \rho u v, \rho v^2 + p, \rho E v + p v)^T,$$



Control volume Γ

$$\frac{d}{dt}\int_{\Gamma} \rho dS + \int_{\partial\Gamma} \rho(\mathbf{U}-\mathbf{K})\cdot \mathbf{n} dl = 0,$$
$$\frac{d}{dt}\int_{\Gamma} \rho \mathbf{U} dS + \int_{\partial\Gamma} [\rho \mathbf{U}\otimes(\mathbf{U}-\mathbf{K}) + pI]\cdot \mathbf{n} dl = 0,$$
$$\frac{d}{dt}\int_{\Gamma} \rho E dS + \int_{\partial\Gamma} [\rho E(\mathbf{U}-\mathbf{K}) + p\mathbf{U}]\cdot \mathbf{n} dl = 0,$$

$$\frac{d}{dt}\int_{\Gamma} \mathbf{W} dS + \int_{\partial\Gamma} [\mathbf{W}(\mathbf{U}-\mathbf{K}) + p\mathbf{H}]\cdot \mathbf{n} dl = 0, \quad H = (0, I, I, \mathbf{U})^T,$$

$$\mathbf{W}_{ij}^{n+1} = \frac{1}{S^{n+1}}[\mathbf{W}_{ij}^n S_{ij}^n - \sum_{k=1}^{4}[\hat{\mathbf{W}}(\hat{\mathbf{U}}-\mathbf{K}) + \hat{p}\hat{\mathbf{H}}]_k \cdot (n_x, n_y)_k L_k \Delta t]$$

Fully-discrete ALE method

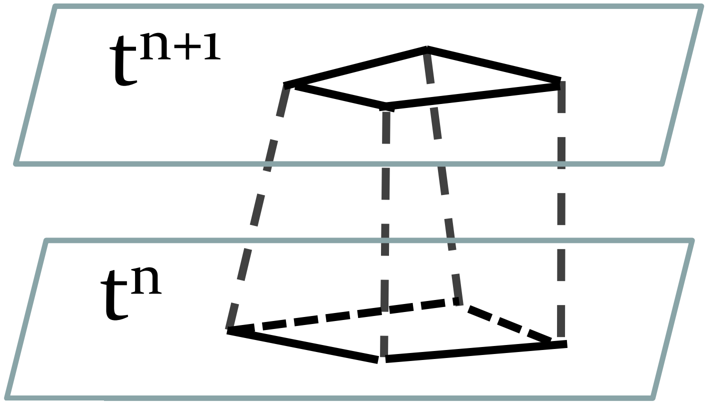
2-D Euler equations

$$W_t + F_x + G_y = 0,$$

$$W = (\rho, \rho u, \rho v, \rho E)^T,$$

$$F = (\rho u, \rho u^2 + p, \rho u v, \rho E u + p u)^T,$$

$$G = (\rho v, \rho u v, \rho v^2 + p, \rho E v + p v)^T,$$



Control volume Ω

$$\int_\Omega \nabla \cdot (W, F, G) dV = \int_{\partial\Omega} (W, F, G) \cdot \boldsymbol{n} dS = 0,$$

$$\int_{\partial\Omega} (W, F, G) \cdot \boldsymbol{n} dS \simeq \sum_{k=1}^{6} (\hat{W}, \hat{F}, \hat{G})_k \cdot (n_t, n_x, n_y)_k S_k,$$

$$W_{ij}^{n+1} S_{ij}^{n+1} - W_{ij}^n S_{ij}^n + \sum_{k=1}^{4} (\hat{W}, F(\hat{W}), G(\hat{W}))_k \cdot (n_t, n_x, n_y)_k S_k = 0,$$

$$W_{ij}^{n+1} = \frac{1}{S^{n+1}} [W_{ij}^n S_{ij}^n - \sum_{k=1}^{4} (\hat{W}, F(\hat{W}), G(\hat{W}))_k \cdot (n_t, n_x, n_y)_k S_k]$$

Comparison of two ALE framework

Semi-discrete method

$$\mathbf{W}_{ij}^{n+1} = \frac{1}{S^{n+1}}[\mathbf{W}_{ij}^{n} S_{ij}^{n} - \sum_{k=1}^{4} [\hat{\mathbf{W}}(\hat{\mathbf{U}} - \mathbf{K}) + \hat{p}\hat{\mathbf{H}}]_k \cdot (n_x, n_y)_k L_k \Delta t]$$

Fully-discrete method

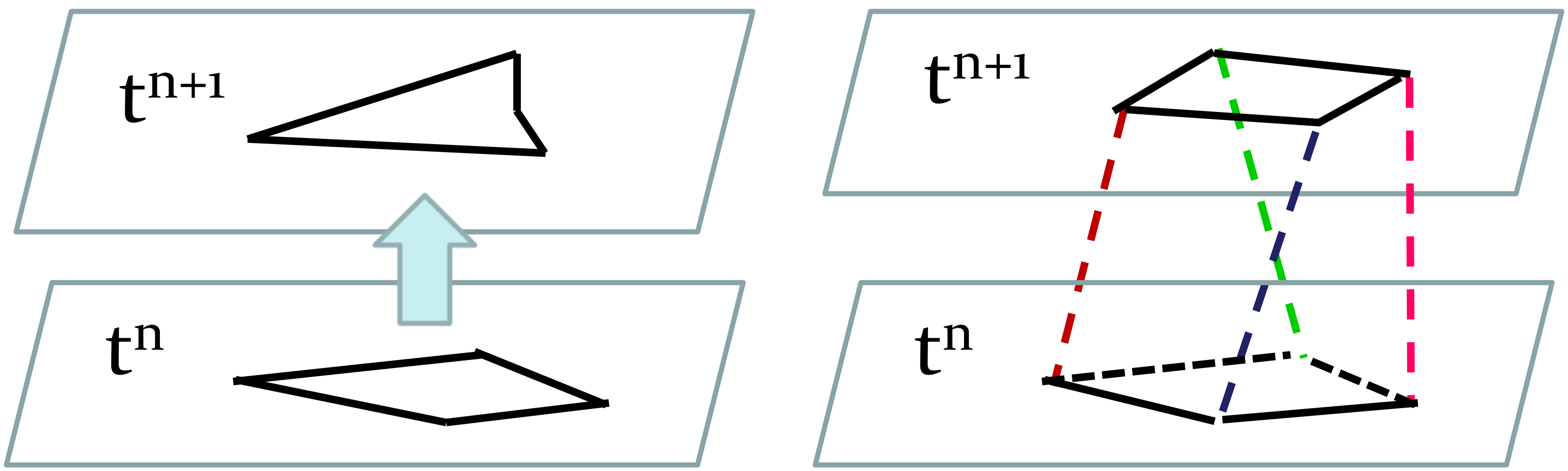
When side is a parallelogram.

$$W_{ij}^{n+1} = \frac{1}{S^{n+1}}[W_{ij}^{n} S_{ij}^{n} - \sum_{k=1}^{4} (\hat{W}, F(\hat{W}), G(\hat{W}))_k \cdot (n_t, n_x, n_y)_k S_k]$$

- Relation of equivalence
- Shape of control volume
  - Spatial control volume: only twist in space
  - Time-space control volume: twist in time or in space

Question

- When mesh is moving , control volume may cause two kinds of deformation.
  - Spatial deformation
  - Time-space deformation



How to quantify and assess time-space deformation?

Untwisted control volume is taken as a start point.

Untwisted control volume

- Aiming at the following features
  - Side is a plane
  - Mesh could self-adaptively move
  - Mesh reflects some characteristics of flow field
- Finding a possible way
  - Vorticity-free mesh generation

Vorticity-free mesh generation

- According to the Helmholtz theorem.

$$\mathbf{U} = \nabla \varphi + \nabla \times \mathbf{A},$$

$$\mathbf{U}_{div} = \nabla \varphi,$$

$$\mathbf{U}_{vor} = \nabla \times \mathbf{A}, \qquad \nabla \cdot \mathbf{A} = 0,$$

- A vector field can be uniquely determined.

$$\mathbf{U} = \mathbf{U}_{div} + \mathbf{U}_{vor},$$

- Avoiding twist, $\mathbf{U}_{div}$ is taken as the mesh velocity K.

$$\mathbf{K} = \mathbf{U}_{div} = \nabla \varphi,$$

- Solving the Laplacian equation to get $\nabla \varphi$.

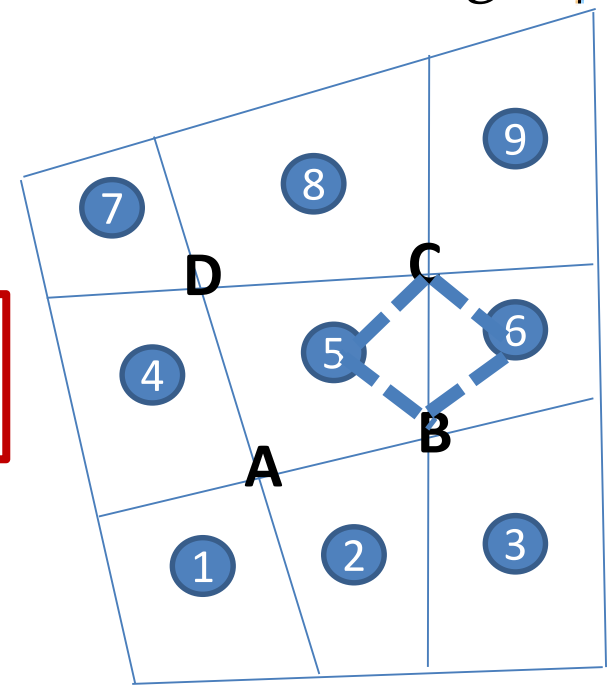
$$\nabla \cdot \nabla \varphi = \nabla \cdot \mathbf{U}_{div}, \quad \mathbf{n} \cdot \nabla \varphi = \mathbf{n} \cdot \mathbf{U}.$$

Laplacian operator's 9-point scheme

- Discretizing Laplacian operator on a nonuniform mesh to get φ .

To solve $\nabla \cdot \nabla \varphi = \nabla \cdot \mathbf{U}_{div}$,

With boundary condition $\mathbf{n} \cdot \nabla \varphi = \mathbf{n} \cdot \mathbf{U}$.



9-point stencil

Integrating over cell,

$$\int \nabla \cdot \nabla \varphi d\tau = \int \nabla \cdot \mathbf{U}_{div} d\tau, \quad \boxed{\int \mathbf{n} \cdot \nabla \varphi dS = \int \mathbf{n} \cdot \mathbf{U}_{div} dS}$$

$\nabla \varphi$ of the cell edge is approximated by

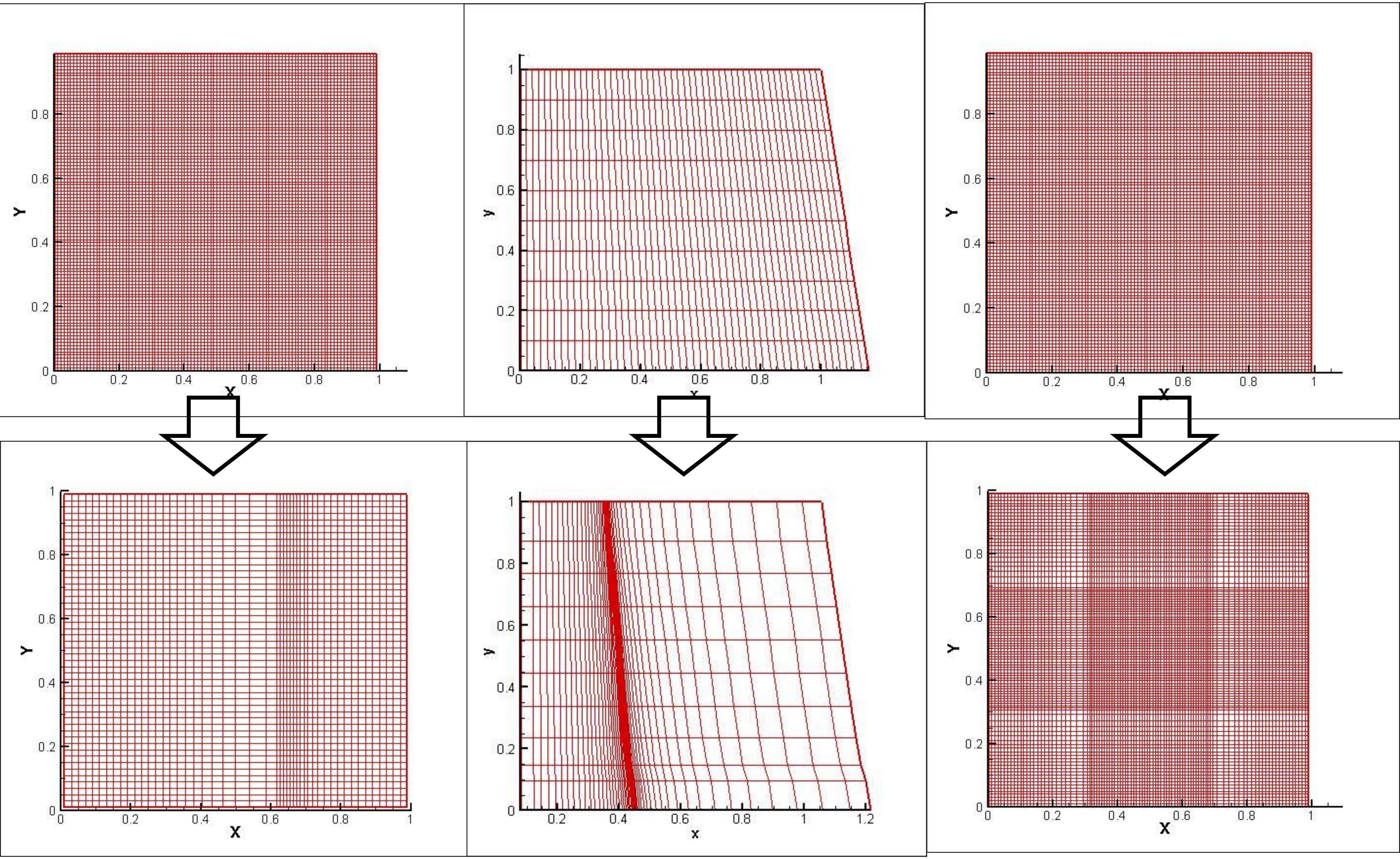
$$\nabla \varphi = \frac{1}{A_{cell}} \int_{cell} \mathbf{n} \varphi d\lambda$$

φ at the vertex is approximated by $\frac{1}{4}\sum_{i=1}^{4} c_i \varphi_i$

So Laplace operator's discretization on cell ABCD: $L_5 \varphi = \sum_{j=1}^{9} a_{5j} \varphi_j$

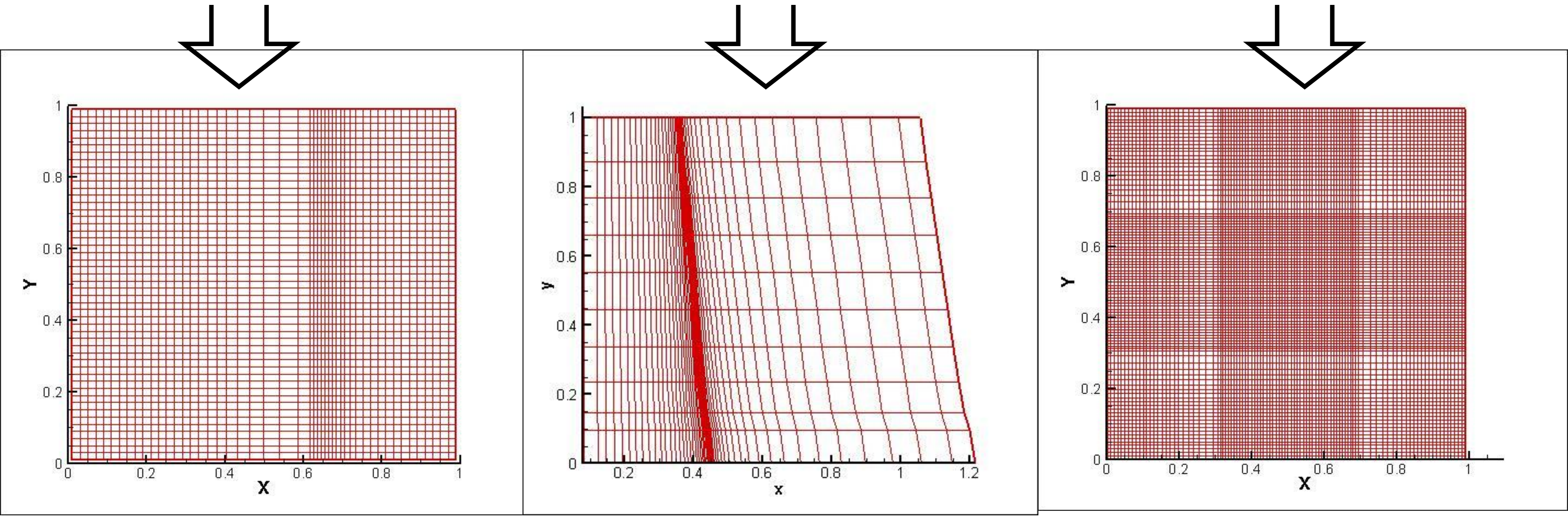
Forming linear equation $\mathbb{A}\varphi = \mathbb{B}$ → ITERATIVE SOLVERS → φ → $\nabla \varphi$.

Vorticity-free mesh



Vorticity-free mesh

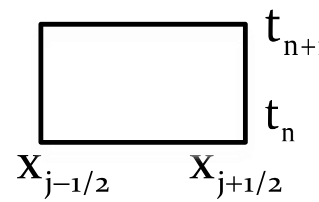
- Such mesh velocity is unique by the Helmholtz decomposition theorem.
- Mesh angle is unchanged.
- The time-space control volume is convex and its side is plane.



A fully-discrete high-order ALE method

- Discrete framework
  - Fully-discrete method
- ALE mesh
  - Vorticity-free mesh
- Flux computation
  - GRP solver

Brief review of GRP method



<u>1-D Euler equations:</u>

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = 0,$$
$$\mathbf{U} = (\rho, \rho u, \rho E)^\top,$$
$$\mathbf{F}(\mathbf{U}) = (\rho u, \rho u^2 + p, u(\rho E + p))^\top,$$

- Discretizing in Eulerian finite volume framework.

$$\mathbf{U}_j^{n+1} = \mathbf{U}_j^n - \frac{\Delta t_n}{\Delta x_j^n}(\hat{\mathbf{F}}_{j+\frac{1}{2}} - \hat{\mathbf{F}}_{j-\frac{1}{2}})$$

- Approximating flux by using the mid-point rule.

.

$$\hat{\mathbf{F}}_{j+\frac{1}{2}} = \mathbf{F}(\mathbf{U}_{j+\frac{1}{2}}^{n+\frac{1}{2}})$$

- Getting $\mathbf{U}_{j+\frac{1}{2}}^{n+\frac{1}{2}}$ by solving the following second order generalized Riemann problem(GRP).

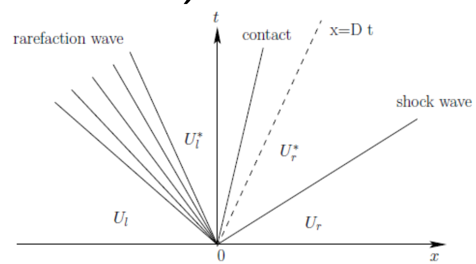
$$\mathbf{U}(x_{j+\frac{1}{2}}, t_n) = \begin{cases} \mathbf{U}_j^n + (\mathbf{U}_x)_j^n (x - x_j^n), & x \in (x_{j-\frac{1}{2}}^n, x_{j+\frac{1}{2}}^n), \\ \mathbf{U}_{j+1}^n + (\mathbf{U}_x)_{j+1}^n (x - x_{j+1}^n), & x \in (x_{j+\frac{1}{2}}^n, x_{j+\frac{3}{2}}^n). \end{cases}$$

GRP solver

GRP solver: the process to get $\mathrm{U}_{j+\frac{1}{2}}^{n+\frac{1}{2}}$ analytically.

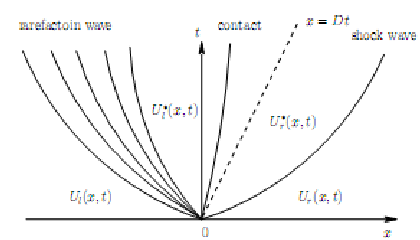
- Firstly solving the associate Riemann problem to get $\mathrm{U}_{j+\frac{1}{2}}^{n}$ by Godunov solver (RP solver).

$$\mathrm{U}(x_{j+\frac{1}{2}},t_n)=\begin{cases}\mathrm{U}_j^n, & x\in(x_{j-\frac{1}{2}}^n,x_{j+\frac{1}{2}}^n),\\ \mathrm{U}_{j+1}^n, & x\in(x_{j+\frac{1}{2}}^n,x_{j+\frac{3}{2}}^n).\end{cases}$$



- Secondly solving a linear system to get $(\frac{\partial \mathrm{U}}{\partial t})_{j+\frac{1}{2}}^{n}$.

$$\begin{cases}a_l(\frac{Du}{D\mathbf{n}})_*+b_l(\frac{Dp}{D\mathbf{n}})_*=d_l,\\ a_r(\frac{Du}{D\mathbf{n}})_*+b_r(\frac{Dp}{D\mathbf{n}})_*=d_r,\end{cases}\Rightarrow\begin{cases}(\frac{Du}{D\mathbf{n}})_*\\ (\frac{Dp}{D\mathbf{n}})_*\end{cases}\Rightarrow(\frac{\partial \mathrm{U}}{\partial t})_{j+\frac{1}{2}}^{n}$$



- At last getting $\mathrm{U}_{j+\frac{1}{2}}^{n+\frac{1}{2}}=\mathrm{U}_{j+\frac{1}{2}}^{n}+\frac{1}{2}(\frac{\partial \mathrm{U}}{\partial t})_{j+\frac{1}{2}}^{n}\Delta t$.

Properties of GRP solver

- An analytic second-order accurate extension of the Godunov solver.
- A close coupling between the spatial and temporal evolution through the analysis of detailed wave interactions .
- A flexibility of application with other method.
  - with adaptive mesh moving method
  - with ALE method
  - …
- A straightforward extension to multidimensional cases .

Flux computation by GRP solver (1)

- Firstly assuming that the data at timestep n are piecewise linear function.

$$\omega(x,y)=\omega(x_c,y_c)+\phi\nabla\omega\cdot(x-x_c,y-y_c)$$

$$\nabla\omega\cong\frac{1}{A}\int_{\Gamma}\mathbf{n}\omega dl\cong\frac{1}{A}\sum_{e\subset\Gamma}|e|\omega_e\mathbf{n}_e,\ \ \omega_e=\frac{\omega_{j,k}+\omega_{neighbor}}{2}.$$

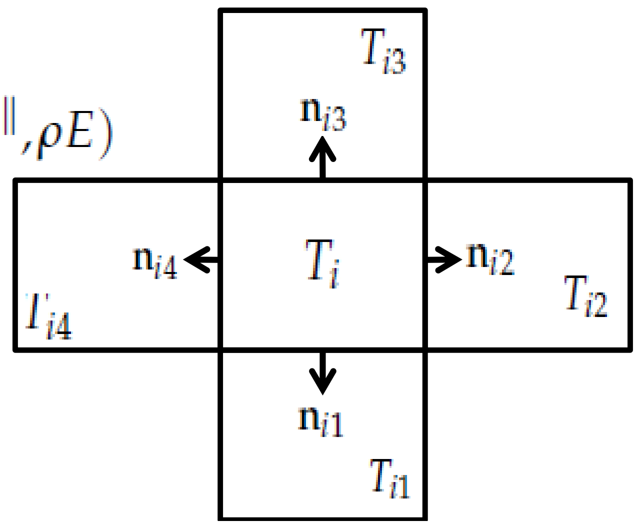
- limiter

$$\phi=min\{1,min_k(\frac{|\omega_k-max_{path}(\omega_k)|}{|\omega_k-max_{cell}(\omega_k)|}),max_k(\frac{|\omega_k-min_{path}(\omega_k)|}{|\omega_k-min_{cell}(\omega_k)|})\}$$

Flux computation by GRP solver (2)

- Secondly forming local 1-D generalized Riemann problems along the outer normal direction at each cell edge.

$$\frac{\partial \mathbf{W}^{\perp}}{\partial t}+\frac{\partial H(\mathbf{W}^{\perp})}{\partial \xi}=0, \quad \mathbf{W}^{\perp}=(\rho,\rho u^{\perp},\rho u^{\parallel},\rho E)$$

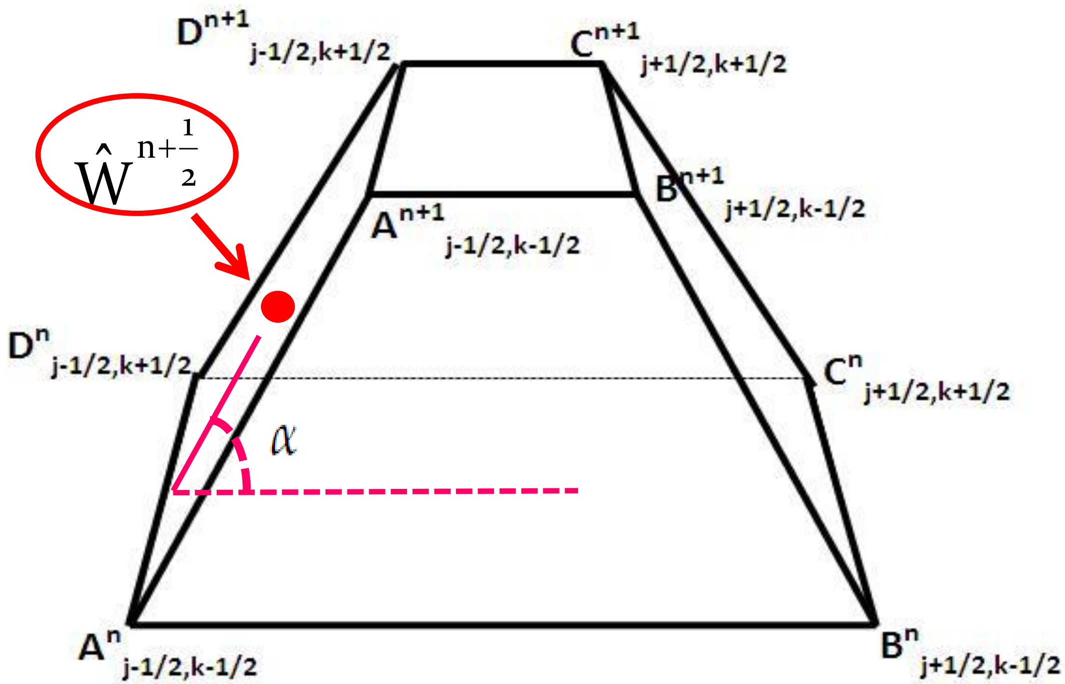


$$\mathbf{W}^{\perp}(0,t^n)=\begin{cases}\mathbf{W}_L^{\perp}+\xi(\mathbf{W}_L^{\perp})'_{\xi}, & \xi<0,\\ \mathbf{W}_R^{\perp}+\xi(\mathbf{W}_R^{\perp})'_{\xi}, & \xi>0,\end{cases}$$

$$\begin{cases}\mathbf{W}_L^{\perp}=\mathbf{W}_{T_i^n}^{\perp}=\frac{1}{|T_i|}\int_{T_i}\mathbf{W}^{\perp}(x,y,t^n)dS,\\ \mathbf{W}_R^{\perp}=\mathbf{W}_{T_{ij}^n}^{\perp}=\frac{1}{|T_{ij}|}\int_{T_{ij}}\mathbf{W}^{\perp}(x,y,t^n)dS,\end{cases} \qquad \begin{cases}(\mathbf{W}_L^{\perp})'_{\xi}=\nabla\mathbf{W}\cdot\mathbf{n}_i,\\ (\mathbf{W}_R^{\perp})'_{\xi}=\nabla\mathbf{W}\cdot\mathbf{n}_{ij},\end{cases}$$

Flux computation by GRP solver (3)

- Computing centroid values $\hat{W}^{n+\frac{1}{2}}$ by GRP solver.
  - Computing the angle between side and bottom for sample procedure in RP and GRP solver
  - Computing $\rho^{n+\frac{1}{2}}, (u^{\perp})^{n+\frac{1}{2}}, p^{n+\frac{1}{2}}$



Flux computation by GRP solver (4)

- Computing tangential velocity $(u^{\parallel})^{n+\frac{1}{2}}$.
  - According to the contact velocity by GRP solver, judging the relative position of the contact and boundary.

$$(u^{\parallel})^{n+\frac{1}{2}} = \begin{cases} u_L^{\parallel} + \frac{\Delta t}{2}(u_L^{\parallel})_t', & u_{contact} > \cot\alpha \\ u_R^{\parallel} + \frac{\Delta t}{2}(u_R^{\parallel})_t', & u_{contact} < \cot\alpha \end{cases}$$

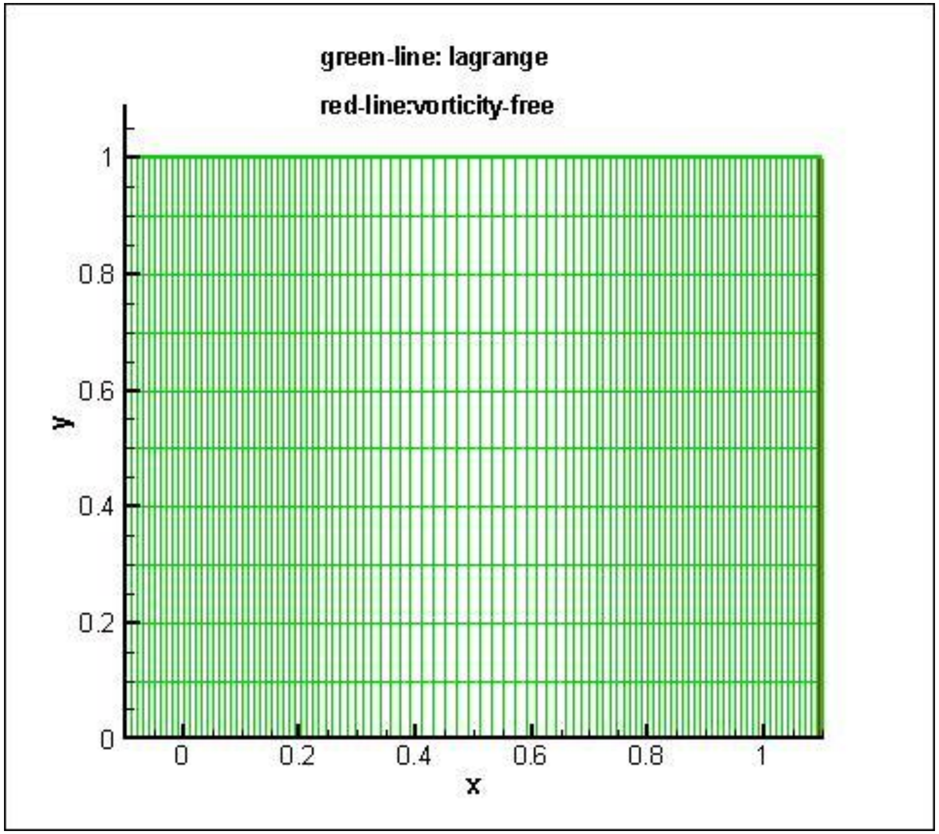
- Transforming $u^{\perp}$ and $u^{\parallel}$ to u and v .
- Computing $\hat{W}^{n+\frac{1}{2}}$.
- Finally approximating the numerical fluxes by using the mid-point rule $\hat{F} = F(\hat{W}^{n+\frac{1}{2}}), \hat{G} = G(\hat{W}^{n+\frac{1}{2}})$.

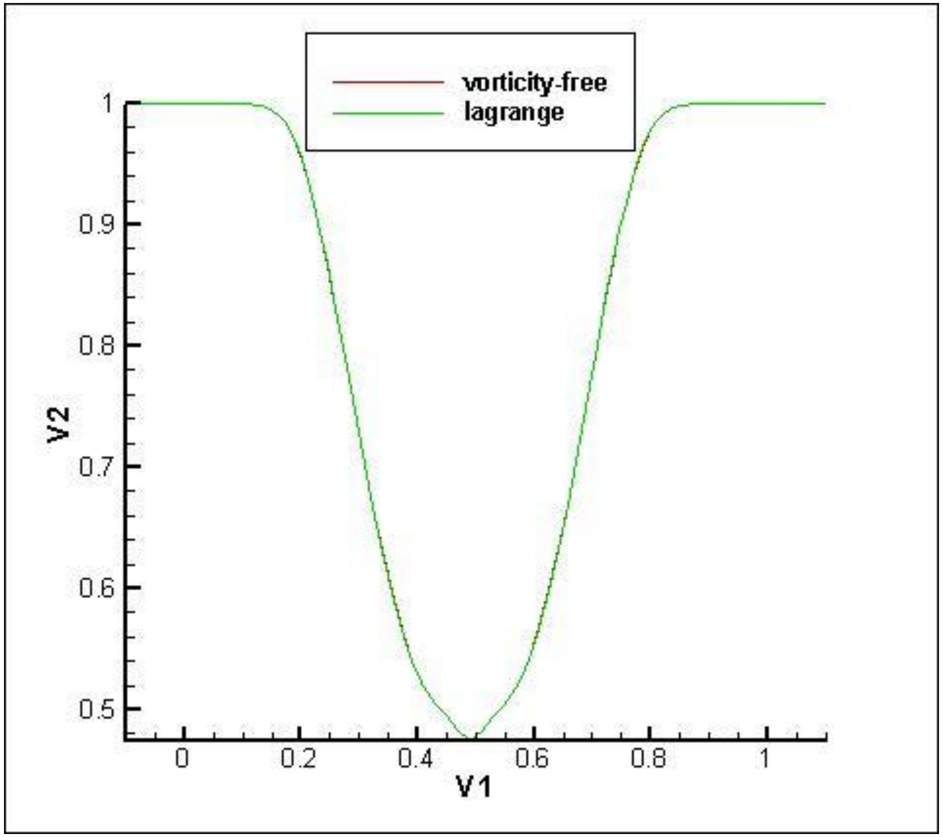
Outline

- ...
- Numerical examples
  - 1-D
  - 2-D
- Concluding remarks

1-D double rarefraction problem

100*100cell

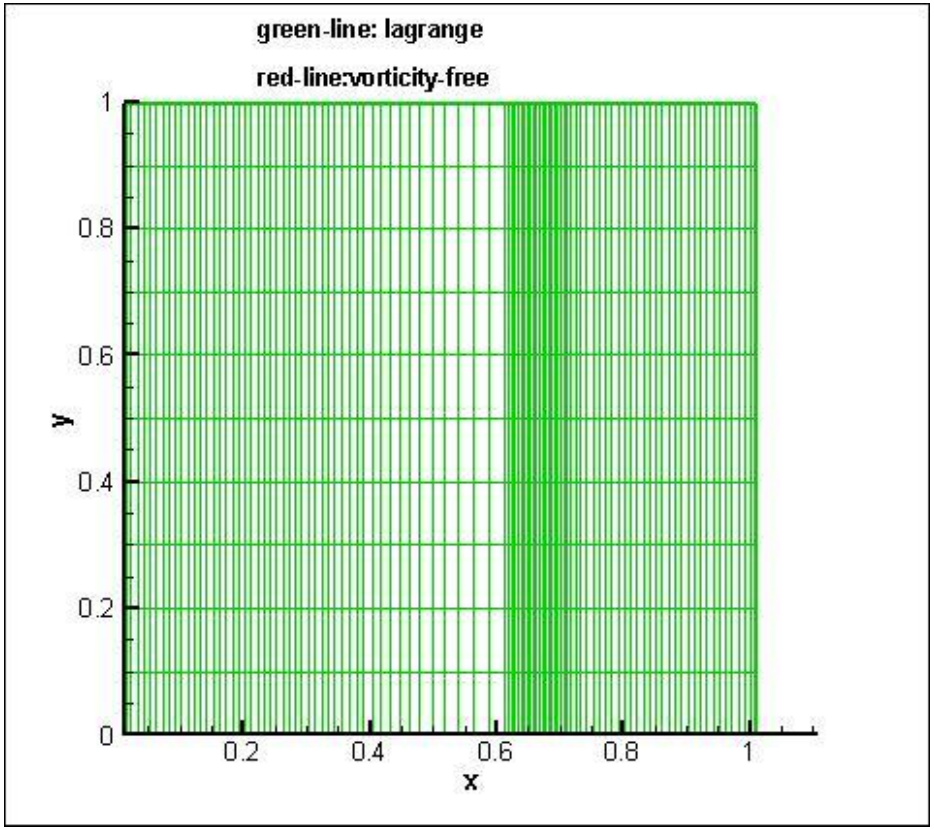


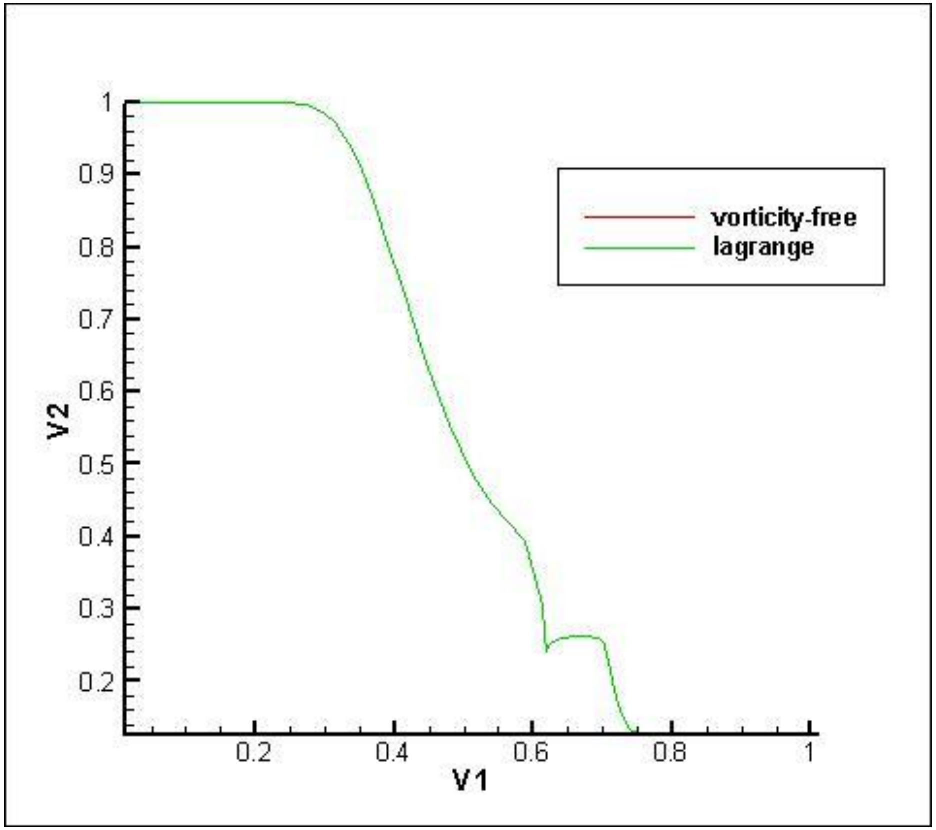


Vorticity-free mesh is the same as Lagrangian mesh.

1-D Sod problem

100*100cell





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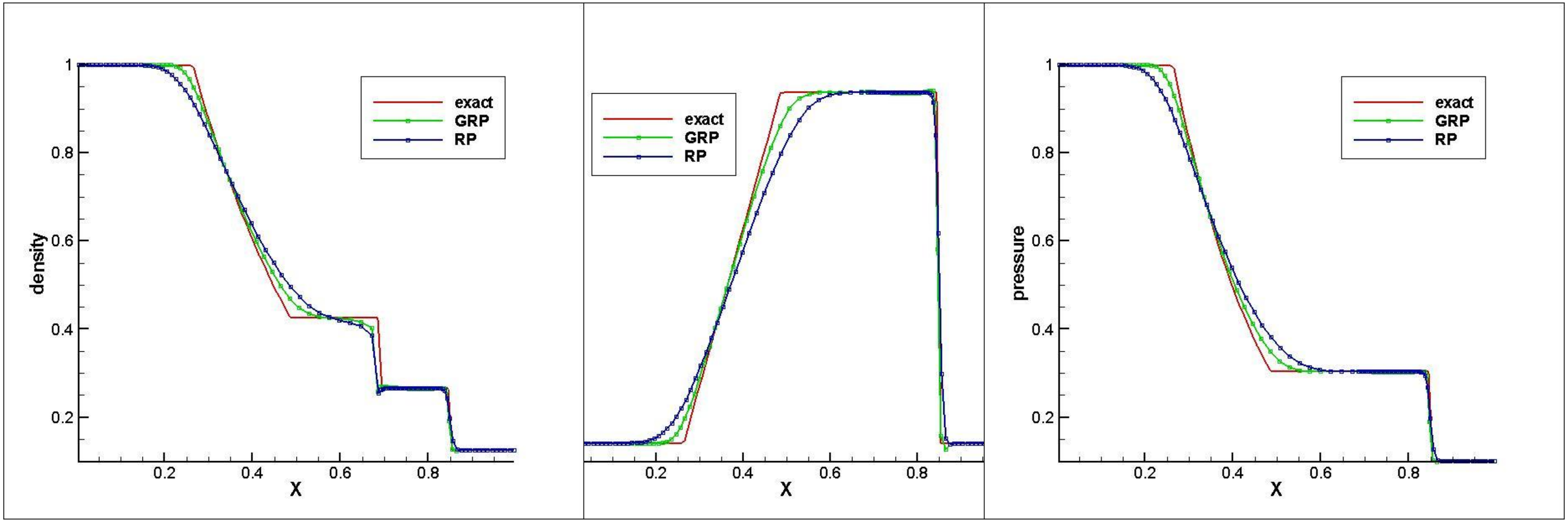
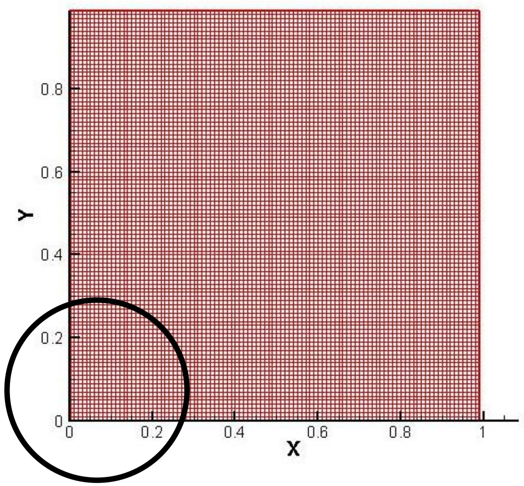


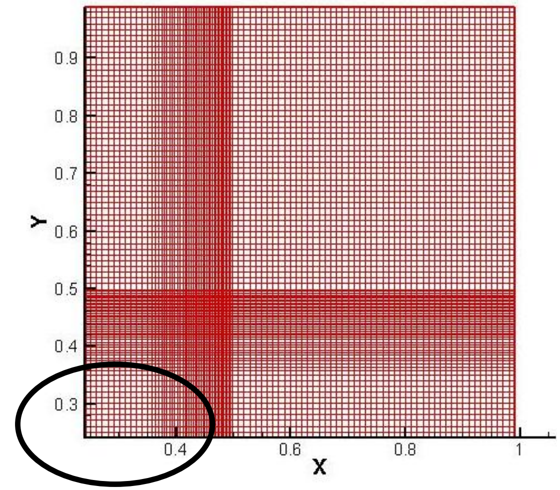
density **velocity** **pressure**

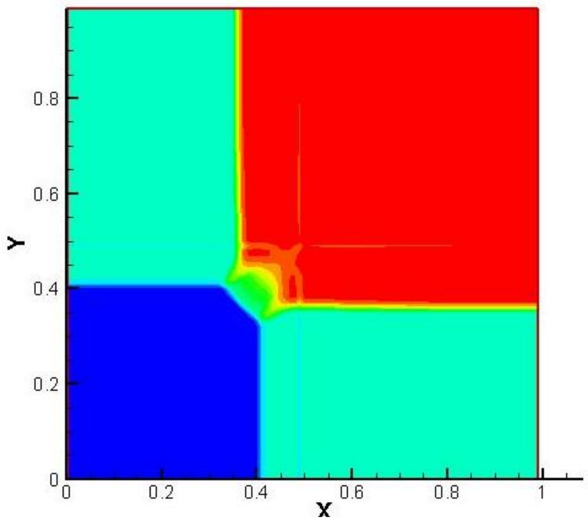
Comparison of RP solver and GRP solver

2-D Riemann problem

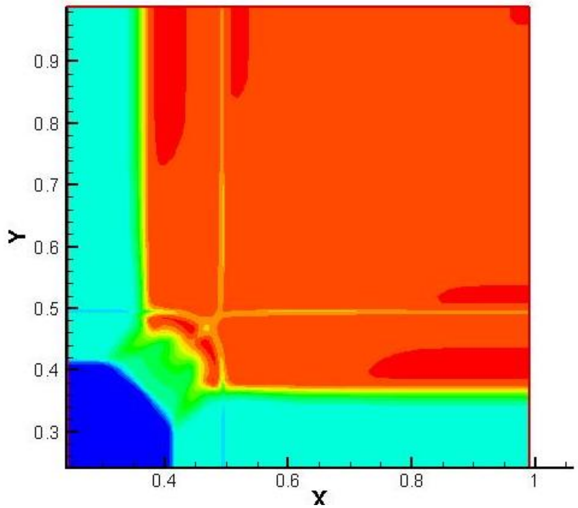


Mesh moving with fluid



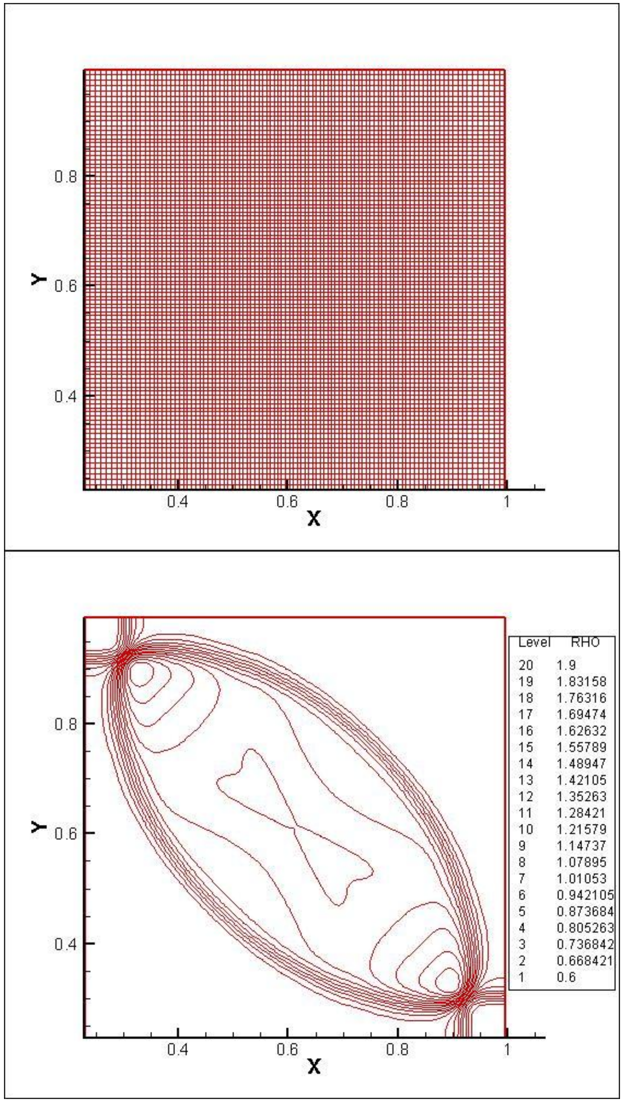


motionless mesh

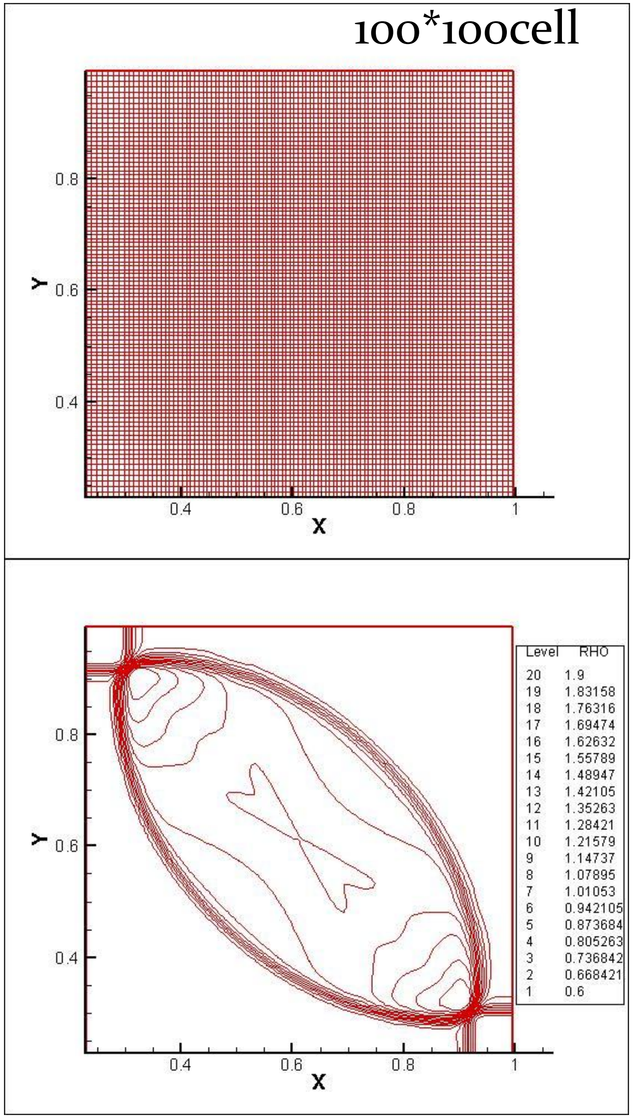


vorticity-free mesh

2-D Riemann problem



vorticity-free mesh+RP



vorticity-free mesh+GRP

Green Vortex Problem

A steady state flow
with the initial conditions:

$$u_0(x,y) = \sin\pi x \cos\pi y,$$

$$v_0(x,y) = -\cos\pi x \sin\pi y,$$

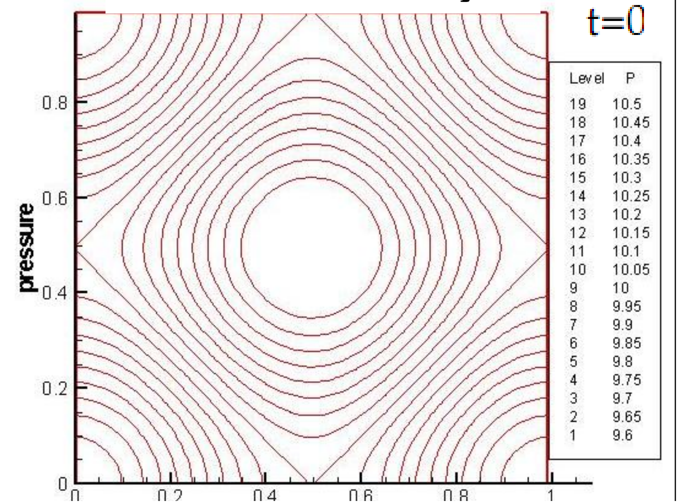
$$\rho_0(x,y) = 1,$$

$$p_0(x,y) = 10 + \frac{1}{4}\rho_0(\cos 2\pi x + \cos 2\pi y),$$

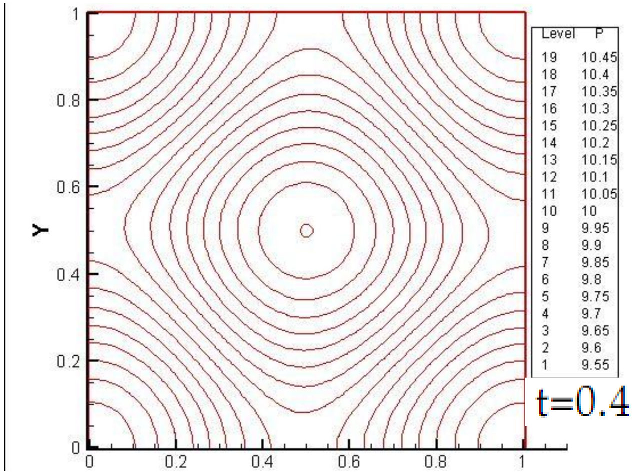
$$e_0(x,y) = \frac{p_0}{\rho_0(\gamma - 1)}, \quad \gamma = \frac{5}{3}.$$

Initial density

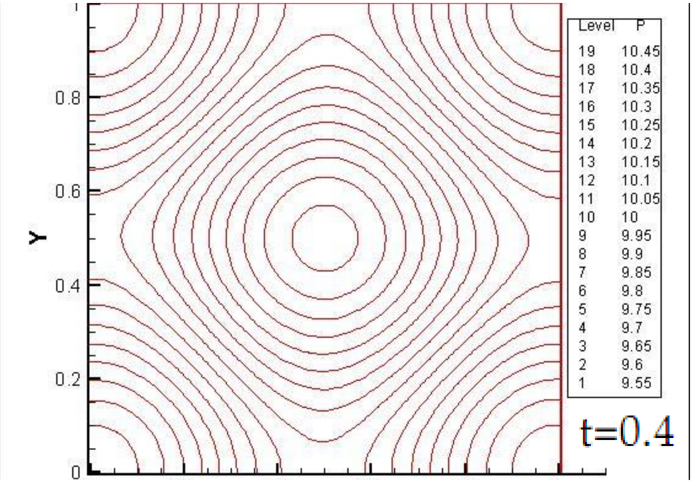
100*100cell
t=0



Vorticity-free mesh is almost the same as Euler mesh.

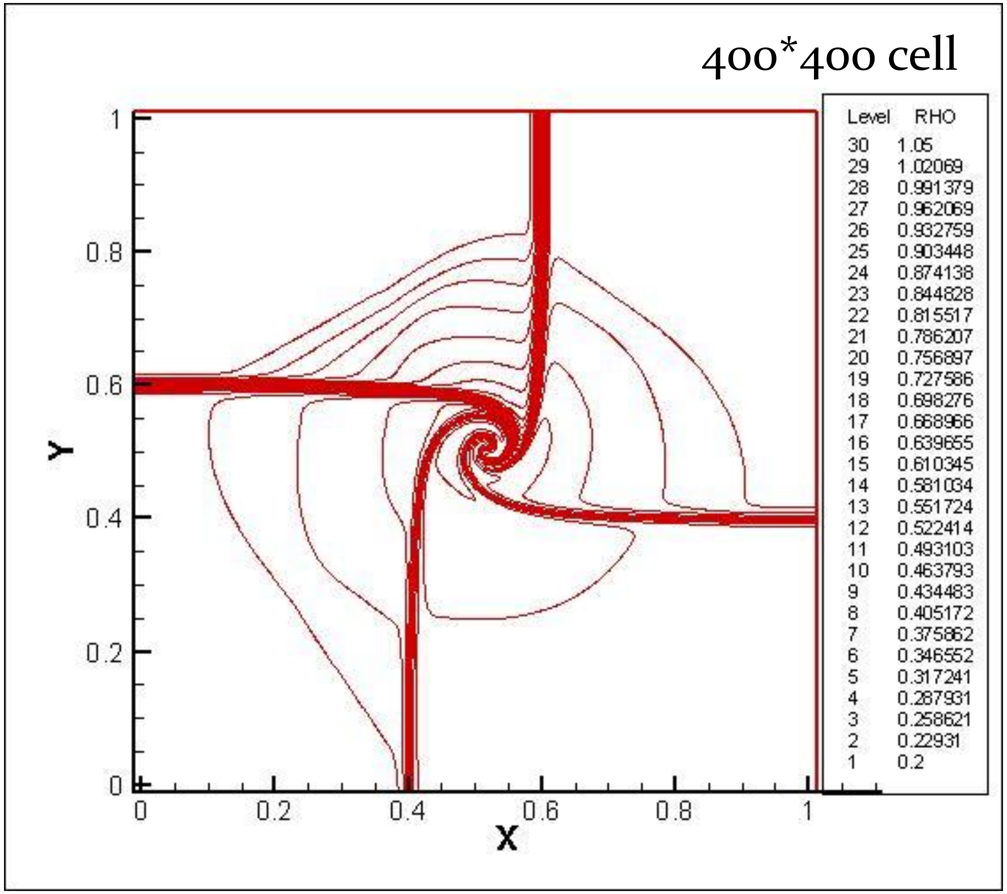


vorticity-free mesh+RP



vorticity-free mesh+GRP

The interaction of vortex sheet and the formation of spiral



vorticity-free mesh+GRP

Vorticity-free mesh is almost the same as Euler mesh.

Concluding remarks

- Vorticity-free mesh was generated to avoid Lagrangian mesh's rotation and keeps control volumes of no twist.
- Numerical fluxes were computed by GRP solver to get a high-precision approximation.
- A remapping-free high-order ALE method was proposed based on the above work.
- Typical numerical examples were tested.

Future work

- More flexible mesh generation method.
- Comprehensive numerical error analysis on twisted time-space control volumes.
- Better reconstruction method and limiter.
- Extension to multi-fluid problems.

Acknowledgement

- Valuable suggestions and discussions with
  - Baolin Tian (IAPCM)
  - Shuanghu Wang (IAPCM)

Thank you for your attention!